

# AN INTRODUCTION TO UNCOUNTABLE CATEGORICITY IN ABSTRACT ELEMENTARY CLASSES

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The aim of these notes is to provide an accessible introduction to abstract elementary classes culminating with the following categoricity theorem: Let  $\mathcal{K}$  be a local abstract elementary class with amalgamation, arbitrarily large models, and countable Löwenheim-Skolem number. If  $\mathcal{K}$  is categorical in  $\aleph_1$  then  $\mathcal{K}$  is categorical in every uncountable cardinal.

Recall that a class  $\mathcal{K}$  of models is said to be *categorical in  $\lambda$*  if all models in  $\mathcal{K}$  of size  $\lambda$  are isomorphic. The content of the above categoricity theorem is that categoricity transfers upwards. The proof does not use any heavy stability theoretic machinery such as prime models or splitting. The class of models of a countable first order theory forms a local abstract elementary class with amalgamation, arbitrarily large models, and countable Löwenheim-Skolem number. In particular, the above theorem gives a simple proof of the upward part of Morley's categoricity theorem.

These notes are separated in a series of chapters centered around an important ingredient so that each chapter contains at least one result interesting in its own right. We sometimes state a more general form of each theorem and prove the particular case in the context of interest. For a fuller introduction to the subject and more precise references, the reader is invited to consult Baldwin's online book [Ba].

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## 0. INTRODUCTION

In this introduction we discuss a more general categoricity theorem proved in [Le1]. We explain the context, give examples, look at how it fits in Shelah's categoricity programme, and outline the proof. In the following chapters we develop all the necessary tools to prove the particular case formulated above ([BaLe2]). The additional assumption of locality allows us to avoid the use of splitting.

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**Theorem 0.1.** *Let  $\mathcal{K}$  be a tame abstract elementary class with amalgamation, arbitrarily large models, and countable  $LS(\mathcal{K})$ . If  $\mathcal{K}$  is categorical in  $\aleph_1$  then  $\mathcal{K}$  is categorical in every uncountable cardinal. More generally, if  $\mathcal{K}$  is categorical in a successor cardinal then  $\mathcal{K}$  is categorical in every cardinal above it.*

Let us start with the definition of abstract elementary class.

**Definition 0.2.** We say that  $(\mathcal{K}, \prec_{\mathcal{K}})$  is an *abstract elementary class* (AEC) if  $\mathcal{K}$  is a class of models in a fixed language  $L$ , partially ordered by the relation  $\prec_{\mathcal{K}}$ , such that:

- (A1)  $(\mathcal{K}, \prec_{\mathcal{K}})$  is closed under isomorphism;
- (A2) If  $M, N \in \mathcal{K}$  and  $M \prec_{\mathcal{K}} N$  then  $M \subseteq N$  i.e.,  $M$  is a submodel of  $N$ ;
- (A3) There is a cardinal  $LS(\mathcal{K})$  such that for all  $N \in \mathcal{K}$  and  $A \subseteq N$  there is  $M \prec_{\mathcal{K}} N$  containing  $A$  of size at most  $|A| + LS(\mathcal{K})$ ;
- (A4) If  $M, N, M^* \in \mathcal{K}$  with  $M \subseteq N$  and  $M, N \prec_{\mathcal{K}} M^*$  then  $M \prec_{\mathcal{K}} N$ ;
- (A5)  $\mathcal{K}$  is closed under Tarski-Vaught chains: Let  $\alpha$  be an ordinal. Let

$$(M_i : i < \alpha)$$

be a  $\prec_{\mathcal{K}}$ -increasing and continuous chain of models in  $\mathcal{K}$ . Then

$$\bigcup_{i < \alpha} M_i \in \mathcal{K}.$$

Also  $M_0 \prec_{\mathcal{K}} \bigcup_{i < \alpha} M_i$  and further, if  $M_i \prec_{\mathcal{K}} N \in \mathcal{K}$ , for each  $i < \alpha$ , then  $\bigcup_{i < \alpha} M_i \prec_{\mathcal{K}} N$ .

We say that  $M$  is a  $\mathcal{K}$ -submodel of  $N$  if  $M \prec_{\mathcal{K}} N$ . In (A1), we mean not only that if  $M_1 \in \mathcal{K}$  and  $M_1 \cong M_2$  then  $M_2 \in \mathcal{K}$ , but also that if  $M_1 \prec_{\mathcal{K}} N_1$  and  $f : N_1 \cong N_2$  then  $f(M_1) \prec_{\mathcal{K}} N_2$ . The cardinal  $LS(\mathcal{K})$  in (A3) is called the *Löwenheim-Skolem number*. It is customary to assume that  $LS(\mathcal{K}) \geq |L| + \aleph_0$  and that  $\mathcal{K}$  has no model of size less than  $LS(\mathcal{K})$ . We will adopt this convention here.

Notice that no condition allows us to go up, except by constructing chains. In particular, we do not assume an upward version of (A3). The set-up of abstract elementary classes is formula-free; it is the notion of  $\mathcal{K}$ -submodel which captures the logical component. For example, (A4) and (A5) are the consequences we need in this context from the Tarski-Vaught test.

- Remark 0.3.**
- (1) Some authors phrase (A3) in the following way: There exists a cardinal  $LS(\mathcal{K})$  such that for each  $N \in \mathcal{K}$  and  $A \subseteq N$  of size at most  $LS(\mathcal{K})$  there is  $M \prec_{\mathcal{K}} N$  containing  $A$  of size  $LS(\mathcal{K})$ . A proof by induction on the size of  $A$  using (A5) easily shows that this apparently weaker phrasing is in fact equivalent.
  - (2) Some authors phrase (A5) by requiring only that  $\mathcal{K}$  be closed under unions of  $\prec_{\mathcal{K}}$ -increasing and continuous sequences of models indexed by cardinals, or even just regular cardinals (including  $\aleph_0$ ). It is easy to see by choosing a cofinal subsequence that this phrasing is also equivalent.

The next proposition shows that  $\mathcal{K}$  is closed under very general unions. We recall some terminology. Let  $(I, <)$  be a well-founded partially ordered set. A family  $(M_s : s \in I)$  of models is a  $\mathcal{K}$ -system if each  $M_s \in \mathcal{K}$  and  $s < t$  implies  $M_s \prec_{\mathcal{K}} M_t$ . A  $\mathcal{K}$ -system is called *directed* if  $I$  is directed *i.e.*, for any  $s_1, s_2 \in I$  there is  $t \in I$  such that  $s_1, s_2 < t$ .

**Proposition 0.4.** *Let  $(M_s : s \in I)$  be a directed  $\mathcal{K}$ -system. Then*

$$\bigcup \{M_s : s \in I\} \in \mathcal{K}.$$

*Also  $M_s \prec_{\mathcal{K}} \bigcup \{M_s : s \in I\}$  for each  $s \in I$ . Further, if  $M_s \prec_{\mathcal{K}} N \in \mathcal{K}$ , for each  $s \in I$ , then  $\bigcup \{M_s : s \in I\} \prec_{\mathcal{K}} N$ .*

*Proof.* We prove this by induction on  $|I|$ . If  $I$  is finite, there is nothing to prove since  $I$  has a maximal element. Let  $I$  be infinite. Consider  $(I_i : i < |I|)$  an increasing and continuous family of well-founded directed suborders of  $I$  of size less than  $|I|$  such that  $\bigcup_{i < |I|} I_i = I$ . Let  $M_i := \bigcup \{M_s : s \in I_i\}$ . By induction hypothesis  $M_i \in \mathcal{K}$ . Furthermore,  $(M_i : i < |I|)$  is  $\prec_{\mathcal{K}}$ -increasing by (A5) (since  $M_s \prec_{\mathcal{K}} M_t$  for each  $s \in I_j$ ) and clearly continuous. Another application of (A5) shows that  $\bigcup_{i < |I|} M_i \in \mathcal{K}$ . But

$$\bigcup_{i < |I|} M_i = \bigcup_{s \in I} M_s,$$

so  $M_s \prec_{\mathcal{K}} \bigcup \{M_s : s \in I\} \in \mathcal{K}$ . Finally, if each  $M_s \prec_{\mathcal{K}} N$ , then each  $M_i \prec_{\mathcal{K}} N$  by induction hypothesis. This implies that  $\bigcup_{i < |I|} M_i \prec_{\mathcal{K}} N$  by (A5) and therefore  $\bigcup \{M_s : s \in I\} \prec_{\mathcal{K}} N$ .  $\square$

We now consider some examples. The reader is referred to [Ho] or [Sh] or [Ba] for definitions and details.

- Examples 0.5.**
- (1) Let  $T$  be a universal (or AE) first order theory in the language  $L$ . Then the class  $\mathcal{K}$  of models of  $T$  with  $\prec_{\mathcal{K}}$  taken as the submodel relation forms an AEC with  $\text{LS}(\mathcal{K}) = |L| + \aleph_0$ .  $\mathcal{K}$  has arbitrarily large models if it has infinite models.
  - (2) The class  $\mathcal{K}$  of models of a first order theory  $T$  in the language  $L$  with  $M \prec_{\mathcal{K}} N$  if  $M$  is an elementary submodel of  $N$  forms an abstract elementary class with  $\text{LS}(\mathcal{K}) = |L| + \aleph_0$ , and  $\mathcal{K}$  has arbitrarily large models if it has infinite models. The class  $\mathcal{K}$  does *not* form an abstract elementary class if  $\prec_{\mathcal{K}}$  is simply submodel as (A5) may fail.
  - (3) Let  $T$  be an AE theory in a language  $L$ . The class  $\mathcal{K}$  of existentially closed models of  $T$  taking  $\prec_{\mathcal{K}}$  as either the submodel or the elementary submodel relation forms an AEC with  $\text{LS}(\mathcal{K}) = |L| + \aleph_0$  and  $\mathcal{K}$  has arbitrarily large models.
  - (4) The class  $\mathcal{K}$  of models of a first order theory  $T$  in the language  $L$  omitting a prescribed set of  $T$ -types with  $\prec_{\mathcal{K}}$  as the elementary substructure relation

- forms an abstract elementary class with  $\text{LS}(\mathcal{K}) = |L| + \aleph_0$ .  $\mathcal{K}$  does not always have arbitrarily large models.
- (5) Similarly, let  $\psi \in L_{\omega_1, \omega}$  and consider a fragment  $\mathcal{A}$  of  $L_{\omega_1, \omega}$  containing  $\psi$ . Then the class  $\mathcal{K}$  of models of  $\psi$  with  $M \prec_{\mathcal{K}} N$  if  $M$  is an  $L_{\mathcal{A}}$ -elementary submodel of  $N$  forms an abstract elementary class with  $\text{LS}(\mathcal{K}) = |\mathcal{A}|$ . It does not always have arbitrarily large models. More generally  $\omega_1$  can be replaced by any uncountable cardinal  $\lambda$  and the resulting class is still an AEC.
- (6) Let  $n$  be an integer above the arity of any relation or function in the language  $L$ . Then the class  $\mathcal{K}$  of models of an  $L^n$ -theory under  $L^n$ -elementary submodel is an abstract elementary class with  $\text{LS}(\mathcal{K}) = |L| + \aleph_0$ . It has arbitrarily large models if it has an infinite model.
- (7) Let  $T$  be a superstable complete first order theory in the language  $L$ . Let  $\lambda$  be a cardinal. The class  $\mathcal{K}$  of  $\lambda$ -saturated models of  $T$  with  $\prec_{\mathcal{K}}$  taken as the elementary submodel relation forms an abstract elementary class with  $\text{LS}(\mathcal{K}) = \lambda + \lambda(T)$ , where  $\lambda(T)$  is the first stability cardinal. It has arbitrarily large models. The class of *saturated* models of  $T$  does not form an abstract elementary class under  $\prec$  in general, as the union of  $(M_i : i < \mu^+)$ , where each  $M_i$  is saturated of size  $\mu$  is not saturated in general. If  $T$  is categorical in some cardinal greater than  $|T|$ , then the class  $\mathcal{K}$  of saturated models forms an abstract elementary class with  $\text{LS}(\mathcal{K}) = |L| + \aleph_0$ .
- (8) Let  $\psi \in L_{\omega_1, \omega}(Q)$ , where  $Q$  is the quantifier ‘there exists uncountably many.’ Let  $\mathcal{A}$  be a countable fragment of  $L_{\omega_1, \omega}(Q)$  containing  $\psi$ . The class  $\mathcal{K}$  of models of  $\psi$  under  $\prec_{\mathcal{A}}$  is generally *not* an abstract elementary class. However,  $\mathcal{K}$  can be made into an abstract elementary class by taking  $M \prec_{\mathcal{K}} N$  to be  $\prec^*$ , where  $M \prec^* N$  if  $M$  is an  $L_{\mathcal{A}}$ -elementary submodel of  $N$  satisfying in addition that if  $M \models \neg Qx\psi(x)$  then there are no new realizations of  $\psi$  in  $N$ . This abstract elementary class has  $\text{LS}(\mathcal{K}) = \aleph_1$ .  $\mathcal{K}$  is not an AEC under  $\prec^{**}$ , where  $M \prec^{**} N$  if  $M \prec^* N$  and in addition if  $M \models Qx\psi(x)$  there then is a realisation of  $\phi(x)$  in  $N \setminus M$  (see [Sh48]).
- (9) Extending these results to larger logics like  $L_{\omega_1, \omega_1}$  or  $L_{\infty, \omega}$  (say with  $\prec_{\mathcal{K}}$  capturing at least  $L_{\infty, \omega}$ -equivalence) does *not* produce AECs in general. Concrete examples of this are well-orders (see the Appendix) and Noetherian rings respectively.
- (10) The class of Banach spaces (in any reasonable language) does *not* form an AEC as unions of Banach spaces are generally not Banach spaces (they may fail to be complete). Considering the class of Banach spaces and their metric subspaces does form an AEC under the notion of subspace for example.
- (11) Let  $L'$  be a language containing  $L$ . Let  $T'$  be a first order theory in the language  $L'$  and  $\Gamma$  a set of  $T'$ -types. The class  $PC(T', L, \Gamma)$  of reducts to  $L$  of models of  $T'$  omitting all types in  $\Gamma$  does *not* form an abstract elementary class in general if  $\prec_{\mathcal{K}}$  is to satisfy (A2), as (A5) may fail. For example, the class of free groups in the language of groups does not form an abstract elementary class under any  $\prec_{\mathcal{K}}$  extending the subgroup relation,

but it does in the language of group with an additional predicate for the set of generators (where it belongs to homogeneous model theory). Let us finally note the following example of Silver: Consider the class  $\mathcal{K}$  of models  $(A, U)$ , where  $U$  is a unary predicate, under the requirement that  $|A| \leq 2^{|U|}$ . This is easily seen to be a  $PC(T', L, \Gamma)$ -class (in fact  $\Gamma$  is empty). Then  $\mathcal{K}$  is categorical in  $\lambda$  if and only if  $\lambda$  is a strong limit. Shelah conjectures (see below) that this sort of behavior cannot happen for AECs.

We now consider the appropriate notion of map in this category.

**Definition 0.6.** Let  $M, N \in \mathcal{K}$ . We say that an  $L$ -embedding  $f : M \rightarrow N$  is a  $\mathcal{K}$ -embedding if in addition  $f(M) \prec_{\mathcal{K}} N$ .

Observe that by (A5) if  $(f_i : M_i \rightarrow N_i \mid i < \alpha)$  is an increasing and continuous sequence of  $\mathcal{K}$ -embeddings then

$$\bigcup_{i < \alpha} f_i : \bigcup_{i < \alpha} M_i \rightarrow \bigcup_{i < \alpha} N_i$$

is a  $\mathcal{K}$ -embedding.

We turn to amalgamation.

**Definition 0.7.** We say that an AEC  $(\mathcal{K}, \prec_{\mathcal{K}})$  has *amalgamation* if it satisfies the following two properties:

- (1) The *amalgamation property* (AP): If  $M_0 \prec_{\mathcal{K}} M_1, M_2 \in \mathcal{K}$  then there is a model  $M^*$  and  $\mathcal{K}$ -embeddings  $f_\ell : M_\ell \rightarrow M^*$  which are the identity on  $M_0$  for  $\ell = 1, 2$ .
- (2) The *joint embedding property* (JEP): If  $M_1, M_2 \in \mathcal{K}$  then there is a model  $M^*$  and  $\mathcal{K}$ -embeddings  $f_\ell : M_\ell \rightarrow M^*$  for  $\ell = 1, 2$ .

In the previous definition, we do not require the images of the models in the amalgam  $M^*$  to be disjoint.

If an abstract elementary class  $\mathcal{K}$  has amalgamation and arbitrarily large models then  $\mathcal{K}$  has no maximal models. The set-up of AEC with amalgamation and arbitrarily large models is essentially the one introduced by Jónsson and Fraïssé (see [BeSI]), and allows us to construct *universal-homogeneous* models. We will make this notion more precise in the sequel.

JEP is not a crucial property: If  $\mathcal{K}$  is an AEC with AP, then  $\mathcal{K}$  can be partitioned in several AEC subclasses satisfying JEP. Moreover, in the categoricity context, a weaker version of JEP always holds (all models of size at most the categoricity cardinal can be jointly embedded) which is enough for most purposes. Finally, if  $\mathcal{K}$  has a prime model over the empty set, then JEP follows from AP. In the following list of examples we focus on AP.

- Examples 0.8.** (1) Let  $T$  be an AE first order theory in the language  $L$ . Then the class  $\mathcal{K}$  of e.c. models of  $T$  with  $\prec_{\mathcal{K}}$  taken as submodel has amalgamation if and only if  $\mathcal{K}$  is a Robinson theory. If  $\prec_{\mathcal{K}}$  is elementary submodel then  $\mathcal{K}$  always has AP.
- (2) The class  $\mathcal{K}$  of models of a first order theory  $T$  in the language  $L$  with  $M \prec_{\mathcal{K}} N$  if  $M$  is an elementary submodel of  $N$ . Then  $\mathcal{K}$  has AP (and JEP when  $T$  is complete).
- (3) All standard Hrushovski style constructions produce AECs with arbitrarily large models and amalgamation.
- (4) The class of models of a first order theory  $T$  in the language  $L$  omitting a prescribed set of types with elementary substructure does not always have amalgamation. But it does when the class contains arbitrarily large homogeneous models (which is equivalent to amalgamation over sets). This is the context of *homogeneous model theory*, which includes the previous examples.
- (5) The class of models of  $\psi \in L_{\omega_1, \omega}$  under  $L_{\mathcal{A}}$ -elementary submodel does not always have amalgamation. It does if  $\mathcal{K}$  is *homogeneous* (see the previous example) or if  $\mathcal{K}$  is *excellent*, a context developed by Shelah (see [Sh87a], [Sh87b], see also [Le1] for an exposition).
- (6) The class of models of an  $L^n$ -theory with  $L^n$ -elementary submodel does not have amalgamation in general, but has been studied by several authors under different amalgamation properties ([Dj], [BaLe1]) both implying amalgamation in our sense.

Let us now consider the problem of *types*. As we pointed out, we do not have formulas and hence no adequate syntactic notion of types. We therefore deal with a semantic notion; we consider a relation  $\sim$  on triples of the form  $(a, M, N)$ , where  $M \prec_{\mathcal{K}} N$  and  $a$  an element of  $N$ . We say that

$$(a_1, M_1, N_1) \sim (a_2, M_2, N_2)$$

if  $M_1 = M_2$  and there exists a model  $M^*$  and  $\mathcal{K}$ -embeddings  $g_\ell : N_\ell \rightarrow M^*$ , for  $\ell = 1, 2$ , such that

$$g_1 \upharpoonright M_1 = id = g_2 \upharpoonright M_2 \quad \text{and} \quad g_1(a_1) = g_2(a_2).$$

It is an exercise using AP to check that  $\sim$  is an equivalence relation on such triples. The equivalence class of  $(a, M, N)$  is the *Galois type of  $a$  over  $M$  (in  $N$ )* and will be denoted by  $\text{tp}(a/M, N)$ . Since we consider no other types, we will simply say ‘type’ for ‘Galois type’ but we may choose to use the full phrase for emphasis. We denote by  $S(M)$  the set of Galois types over  $M$ . We say that  $N'$  *realizes*  $\text{tp}(a/M, N)$  if  $M \prec_{\mathcal{K}} N'$  and there exists  $a' \in N'$  such that  $\text{tp}(a'/M, N') = \text{tp}(a/M, N)$ . We also write  $\text{tp}(a/M, N) \upharpoonright M'$  for  $M' \prec_{\mathcal{K}} M$  for  $\text{tp}(a/M', N)$ . Notice finally that under AP if  $p \in S(M)$  and  $M \prec_{\mathcal{K}} N$  then  $p$  has an extension in  $S(N)$ .

In the familiar case where the AEC is the class of models of a first order theory, then Galois types correspond to the usual notion of types over a model defined by a set of formulas. The same holds in the context of homogeneous model theory and in excellent classes, where this is actually a theorem and a crucial consequence of excellence (see [Le1] and [GrKo]).

In general, it is not clear that if  $\text{tp}(a/N) \neq \text{tp}(b/N)$  then there necessarily exists a small  $M \prec_{\mathcal{K}} N$  such that  $\text{tp}(a/M) \neq \text{tp}(b/M)$ . This is where tameness comes in.

**Definition 0.9.** An abstract elementary class  $(\mathcal{K}, \prec_{\mathcal{K}})$  with amalgamation is *tame* if whenever  $\text{tp}(a/N) \neq \text{tp}(b/N)$  for  $N \in \mathcal{K}$  then there is  $M \prec_{\mathcal{K}} N$  of size  $\text{LS}(\mathcal{K})$  such that  $\text{tp}(a/M) \neq \text{tp}(b/M)$ .

We have now defined all the terms in the main theorem. We now discuss how the result fits with respect to other categoricity results in abstract elementary classes. The main conjecture of the subject is due to Shelah:

**Conjecture 0.10** (Shelah). Let  $\mathcal{K}$  be an abstract elementary class. If  $\mathcal{K}$  is categorical in some cardinal above the Hanf number, then  $\mathcal{K}$  is categorical everywhere above the Hanf number.

Note that it is not obvious, a priori, that there is a *Hanf number* for abstract elementary classes, namely a smallest cardinal  $H(\kappa)$  such that any abstract elementary class with  $\text{LS}(\mathcal{K}) \leq \kappa$  which contains a model of size  $H(\kappa)$  contains arbitrarily large models. We prove that the Hanf number exists and is equal to  $\beth_{(2^{\text{LS}(\mathcal{K})})^+}$  in the next chapter.

Shelah considers this conjecture the most important problem in model theory [Sh702]. It is still wide open in general, but he made startling progress towards proving it under the additional assumption of amalgamation ([Sh394]):

**Theorem 0.11** (Shelah). *Let  $\mathcal{K}$  be an abstract elementary class with amalgamation. Assume that  $\mathcal{K}$  is categorical in a successor cardinal  $\lambda^+$  above the second Hanf number. Then  $\mathcal{K}$  is categorical in every cardinal between the first Hanf number and  $\lambda^+$ .*

The second Hanf number is  $H(H(\text{LS}(\mathcal{K})))$  i.e., is obtained by plugging the value  $\beth_{(2^{\text{LS}(\mathcal{K})})^+}$  instead of  $\text{LS}(\mathcal{K})$  in  $\beth_{(2^{\text{LS}(\mathcal{K})})^+}$ . Shelah's proof uses the appropriate notion of saturated models, minimal types, and the absence of Vaughtian pairs. It uses an omitting type theorem to go down to the Hanf number and then goes up inductively. The induction stops at  $\lambda^+$ , as Shelah is unable to show a version of tameness for over models of size  $\lambda^+$  and above. Grossberg and VanDieren assume tameness and using the same proof techniques show ([GrVD]):

**Theorem 0.12** (Grossberg-VanDieren). *Let  $\mathcal{K}$  be a tame abstract elementary class with amalgamation and arbitrarily large models. If  $\mathcal{K}$  is categorical in  $\lambda$  and  $\lambda^+$ , for  $\lambda > \text{LS}(\mathcal{K})$ , then  $\mathcal{K}$  is categorical everywhere above  $\lambda^+$ .*

Note that the notion of tameness that they use is weaker than the one we defined here, it simply require that some model of size less than  $\lambda$  be able to separate two types over any large model.

There are three main differences with our result. The first is that we can prove upward categoricity from a single cardinal. Secondly, we can start at  $\aleph_1$  (Grossberg and VanDieren's result buys upwards categoricity from  $\aleph_1$  and  $\aleph_2$  in the lowest case). In practice, for example for the pseudo-analytic structures studied by Zilber [Zi] and Gavrilovich [Ga], categoricity in  $\aleph_1$  can be established by algebraic or geometric means. However, categoricity in a higher cardinal, even  $\aleph_2$ , is as difficult to establish as the general categoricity problem. The third difference is that we assume that  $\text{LS}(\mathcal{K})$  is countable. We introduce two new ingredients: *quasiminimal types* and *superlimits*. Quasiminimal types allow us to prove categoricity from a single cardinal, and superlimits allow us to start at  $\aleph_1$ . More recently, Grossberg and VanDieren were able to use similar techniques to show upward categoricity from a single successor cardinal for uncountable  $\text{LS}(\mathcal{K})$  also, but they need to start at or above  $\text{LS}(\mathcal{K})^{++}$ .

Together, these results give the following approximation of Shelah's conjecture:

**Corollary 0.13.** *Let  $\mathcal{K}$  be a tame abstract elementary class with amalgamation. If  $\mathcal{K}$  is categorical in a successor cardinal above the Hanf number then  $\mathcal{K}$  is categorical everywhere above the Hanf number.*

We finish this introduction with an outline of the proof of the main theorem in the case when  $\lambda^+ = \aleph_1$ . Other than the two ingredients we mentioned, all the basic ideas are due to Shelah. We consider saturated models with respect to Galois types. We prove that two saturated models of the same size are isomorphic (Chapter 2). The idea is then to show that categoricity in  $\aleph_1$  implies that all the uncountable models are saturated. This is done by showing by induction on  $\mu \geq \aleph_1$  that all models of size  $\mu$  are saturated. The only difficult cases are for  $\mu = \aleph_1$  and  $\mu$  a successor cardinal. A crucial tool is that uncountable categoricity implies  $\aleph_0$ -stability (Chapter 1). This implies the existence of saturated models of size  $\aleph_1$  (Chapter 2), taking care of the base case. To prove the successor case, we will use  $\aleph_0$ -stability again to show the existence of certain types, called quasiminimal (Chapter 5), and certain kind of models, called superlimits (Chapter 3), which can be used to show that there are no Vaughtian pairs (Chapter 6). The full argument is presented in Chapter 7. Chapter 8 contains an appendix with a direct argument on the undefinability of well order adapted from [ShVä].

## 1. SHELAH'S PRESENTATION THEOREM AND STABILITY

We start this chapter with Shelah's Presentation Theorem for abstract elementary classes which is a key tool to understand AECs. The theorem states that every AEC can be represented as a  $PC(T', L, \Gamma)$ -class (recall, however, that not every  $PC(T', L, \Gamma)$ -class is an AEC).

**Theorem 1.1** (Shelah). *Let  $(\mathcal{K}, \prec_{\mathcal{K}})$  be an abstract elementary class in the language  $L$  ( $|L| + \aleph_0 \leq \text{LS}(\mathcal{K})$ ). There exist a language  $L'$  expanding  $L$  of size  $\text{LS}(\mathcal{K})$ , a first order theory  $T'$  in the language  $L'$ , and a set of  $T'$ -types  $\Gamma$  such that*

$$\mathcal{K} = PC(T', L, \Gamma) = \{M' \upharpoonright L : M' \models T' \text{ and omits all types in } \Gamma\}.$$

Moreover, if  $M', N' \models T'$  omit all types in  $\Gamma$  and  $M' \prec N'$  then

$$M' \upharpoonright L \prec_{\mathcal{K}} N' \upharpoonright L.$$

The class  $\mathcal{K}$  of groups in the language of groups with  $\prec_{\mathcal{K}}$  as the subgroup relation shows that the converse of the moreover must fail: Choose  $G \leq H$  such that  $G$  is not an elementary subgroup of  $H$ .

Before proving the theorem, we point out the surprising corollary that abstract elementary classes have a Hanf number. The proof is immediate using the previous theorem and the Hanf number for PC-classes (for computation of Hanf numbers see [Sh]). The central idea in the computation of the Hanf number is the undefinability of well orders. We give a simple argument of the undefinability of well orders of size  $(2^{\text{LS}(\mathcal{K})})^+$  in the context of abstract elementary classes in the Appendix.

**Corollary 1.2** (Shelah). *The Hanf number for AECs is  $\beth_{(2^{\text{LS}(\mathcal{K})})^+}$ .*

We now prove Shelah's Presentation Theorem.

*Proof of the theorem:* Let  $L'$  be the language obtained from  $L$  by adding new  $n$ -ary function symbols  $F_i^n$ , for each  $n < \omega$  and  $i < \text{LS}(\mathcal{K})$ . We let the theory  $T'$  state that the models are nonempty and that for each  $n < \omega$  we have

$$\forall x_0 \dots x_{n-1} F_i^n(x_0, \dots, x_{n-1}) = x_i, \quad \text{for } i < n.$$

We make no assumption on the values of  $F_i^n$  for  $i \geq n$ .

Let  $M \models T'$  and let  $\bar{a} \in M$ . For each subsequence  $\bar{b}$  of  $\bar{a}$ , consider the following subset of  $M$ :

$$U_{\bar{b}} := \{F_i^m(\bar{b}) \mid i < \text{LS}(\mathcal{K})\}, \quad \text{where } \ell(\bar{b}) = m.$$

The set  $U_{\bar{b}}$  depends only on  $\bar{b}$  and  $M$ . In general, the set  $U_{\bar{b}}$  may not be the universe of a submodel of  $M \upharpoonright L$ , but this information is entirely determined by

$$\text{q.f.-tp}_{L'}(\bar{a}/\emptyset, M'),$$

the quantifier free  $L'$ -type of  $\bar{a}$  over the empty set in  $M'$  (in fact, this information is contained in q.f.- $\text{tp}_{L'}(\bar{b}/\emptyset, M')$  already). Suppose that  $U_{\bar{b}}$  is the universe of a submodel of  $M \upharpoonright L$ , then this submodel is entirely determined by  $M$  and  $\bar{b}$  and will be denoted by  $M_{\bar{b}}$ . Then the isomorphism-type of  $M_{\bar{b}}$  and whether it belongs to  $\mathcal{K}$  is also determined by q.f.- $\text{tp}_{L'}(\bar{a}/\emptyset, M')$ . Finally, since  $(\mathcal{K}, \prec_{\mathcal{K}})$  is closed under isomorphism, whether, in addition,  $M_{\bar{b}} \prec_{\mathcal{K}} M_{\bar{a}}$  for some subsequence  $\bar{b}$  of  $\bar{a}$ , is also determined by q.f.- $\text{tp}(\bar{a}/\emptyset, M')$ .

We now define  $\Gamma$ : For each  $M' \models T'$  and each tuple  $\bar{a} \in M'$  we put q.f.- $\text{tp}(\bar{a}/\emptyset, M') \in \Gamma$  unless we have the following two conditions:

- (1) For each subsequence  $\bar{b} \leq \bar{a}$  the set  $U_{\bar{b}}$  is the universe of a submodel of  $M' \upharpoonright L$ , denoted by  $M_{\bar{b}}$ , and  $M_{\bar{b}} \in \mathcal{K}$ .
- (2) For each subsequence  $\bar{b} \leq \bar{a}$  we have  $M_{\bar{b}} \prec_{\mathcal{K}} M_{\bar{a}}$ .

We now show that  $\mathcal{K} = PC(T', L, \Gamma)$ :

First, assume that  $M \in PC(T', L, \Gamma)$ . Then there is  $M' \models T'$  omitting all the types in  $\Gamma$  such that  $M = M' \upharpoonright L$ . Consider the sets  $U_{\bar{a}}$ , for each  $\bar{a} \in M$ . Since  $M'$  omits all the types in  $\Gamma$ , each set  $U_{\bar{a}}$  is the universe of a submodel  $M_{\bar{a}}$  of  $M = M' \upharpoonright L$  such that  $M_{\bar{a}} \in \mathcal{K}$ . Moreover, the  $\mathcal{K}$ -system

$$(M_{\bar{a}} : \bar{a} \in M)$$

is directed since  $M_{\bar{a}}, M_{\bar{b}} \prec_{\mathcal{K}} M_{\bar{a}\bar{b}}$ . Hence by Proposition 0.4 we have

$$\bigcup \{M_{\bar{a}} : \bar{a} \in M\} \in \mathcal{K}.$$

But since  $M' \models T'$ , we have  $\bar{a} \in U_{\bar{a}}$  and so  $\bigcup_{\bar{a} \in M} M_{\bar{a}} = M$ . This shows that  $M \in \mathcal{K}$ .

For the converse, assume that  $M \in \mathcal{K}$ . We will define the expansion  $M'$  of  $M$  by assigning values to the functions  $\{F_i^n : i < \text{LS}(\mathcal{K})\}$  by induction on  $n < \omega$  in such a way that for each  $\bar{a} \in M$  of length  $n$  the set  $U_{\bar{a}} = \{F_i^n(\bar{a}) \mid i < \text{LS}(\mathcal{K})\}$  can be made into a  $\mathcal{K}$ -submodel of  $M$ , denoted by  $M_{\bar{a}}$ , and the assignment is consistent with  $T'$ .

For  $n = 0$  choose  $M_{\emptyset} \prec_{\mathcal{K}} M$  of size  $\text{LS}(\mathcal{K})$  and let  $U_{\emptyset} = \{F_i^0 \mid i < \text{LS}(\mathcal{K})\}$  be an enumeration of the universe of  $M_{\emptyset}$ . Having done this for  $n$ , let  $\bar{a} \in M$  of length  $n + 1$ . Choose  $M_{\bar{a}} \prec_{\mathcal{K}} M$  of size  $\text{LS}(\mathcal{K})$  containing the set  $\bigcup \{U_{\bar{b}} : \bar{b} < \bar{a}\}$ , which contains  $\bar{a}$ , and let  $U_{\bar{a}} = \{F_i^n(\bar{a}) \mid i < \text{LS}(\mathcal{K})\}$  be an enumeration of the universe of  $M_{\bar{a}}$  such that  $F_i^n(\bar{a}) = a_i$ , for  $i < n + 1$ .

The induction is complete and determines a unique expansion  $M'$  of  $M$ . It is easy to see that  $M' \models T'$  and omits every type in  $\Gamma$ .

The ‘moreover’ is clear also: Let  $M \prec N$  be models of  $T'$  omitting all the types in  $\Gamma$ . Then for  $\bar{a} \in M$ , we have  $M_{\bar{a}} = N_{\bar{a}}$  in the obvious notation. Hence,

$$M \upharpoonright L = \bigcup \{M_{\bar{a}} : \bar{a} \in M\} = \bigcup \{N_{\bar{a}} : \bar{a} \in M\} \prec_{\mathcal{K}} \bigcup \{N_{\bar{a}} : \bar{a} \in N\} = N \upharpoonright L$$

by using the definitions and Proposition 0.4 again.  $\square$

We finish this chapter with Shelah’s notion of *stability*, and more particularly  $\aleph_0$ -stability. As in the first order case, it is one of the central tools to study categoricity. Recall that if  $\mathcal{K}$  is an AEC with amalgamation, we have a notion of Galois types, and for each  $M \in \mathcal{K}$ , the set  $S(M)$  is the set of Galois types over  $M$ .

**Definition 1.3.** Let  $(\mathcal{K}, \prec_{\mathcal{K}})$  be an AEC with amalgamation. We say that  $\mathcal{K}$  is  $\mu$ -stable if for each  $M \in \mathcal{K}$  of size  $\mu$  we have  $|S(M)| \leq \mu$ .

Another important consequence of Shelah’s Presentation Theorem is the possibility to represent models of  $\mathcal{K}$  as *Ehrenfeucht-Mostowski models*, provided that  $\mathcal{K}$  has arbitrarily large models. This is used in the proof of the next theorem. The proof is conceptually completely similar to its first order counterpart. The only difference is that we have to consider Galois types instead of first order types. This is not a significant change, as all it needs is to work with sufficiently homogeneous orders, so that equality of first order types corresponds to equality of orbits in the order.

**Theorem 1.4 (Shelah).** *Let  $(\mathcal{K}, \prec_{\mathcal{K}})$  be an AEC with amalgamation and arbitrarily large models. If  $\mathcal{K}$  is categorical in  $\lambda > \text{LS}(\mathcal{K})$  then  $\mathcal{K}$  is  $\mu$ -stable for each  $\mu$  such that  $\text{LS}(\mathcal{K}) \leq \mu < \lambda$ .*

*Proof.* We prove the case where  $\text{LS}(\mathcal{K}) = \aleph_0$  and  $\lambda = \aleph_1$  but the general case is similar. We must show that  $\mathcal{K}$  is  $\aleph_0$ -stable.

Suppose, for a contradiction, that  $\mathcal{K}$  is not  $\aleph_0$ -stable. Let  $M$  be a countable model such that  $S(M)$  is uncountable. By amalgamating over  $M$  all the models containing realizations of types in  $S(M)$ , we can find an uncountable model  $N \in \mathcal{K}$ , with  $M \prec_{\mathcal{K}} N$  such that  $N$  realizes uncountably many types over  $M$ . By using (A3) if necessary, we may assume that  $N$  has size  $\aleph_1$ .

Now  $\mathcal{K} = PC(T', L, \Gamma)$  by Shelah’s Presentation Theorem. As in the first order case, we may assume that  $L'$  contains Skolem functions. Since  $\mathcal{K}$  has arbitrarily large models, we have Ehrenfeucht-Mostowski models. Consider  $N' = EM(\omega_1) \upharpoonright L \in \mathcal{K}$ . Then  $N'$  has size  $\aleph_1$ . We claim that  $N'$  realizes only countably many Galois types over each countable submodel  $M'$ : Let  $M'$  be given and let  $J \subseteq I$  be countable such that  $M' \subseteq EM(J) \upharpoonright L$ . First,  $(\omega_1, <)$  is atomically  $\aleph_0$ -stable, that is, there are only countably many cuts in  $\omega_1$  over  $J$ . Secondly, if two finite sequences  $a, b \in \omega_1$  realize the same cut over  $J$ , there exists  $(I', <')$  extending  $(\omega_1, <)$  such that  $a$  and  $b$  are automorphic over  $J$  in the order  $I'$ . Hence

$a$  and  $b$  induce the same Galois type over  $EM(J) \upharpoonright L$  (and hence over  $M'$ ) in  $N'$ .  
(Note that  $I'$  as above can be chosen to be  $\omega_1^{<\omega}$ .)

But, by categoricity in  $\aleph_1$  we have  $N \cong N'$ , which is a contradiction.  $\square$

## 2. SATURATED MODELS: EXISTENCE AND UNIQUENESS

In this chapter, we assume that  $(\mathcal{K}, \prec_{\mathcal{K}})$  is an abstract elementary class with amalgamation and arbitrarily large models. We introduce saturated models which are the backbone of the proof of the categoricity theorem.

**Definition 2.1.** We say that  $N \in \mathcal{K}$  is  $\mu$ -saturated if  $N$  realizes each  $p \in S(M)$  for  $M \prec_{\mathcal{K}} N$  of size less than  $\mu$ . We say that  $N$  is saturated if it is  $\|N\|$ -saturated.

We first prove that stability implies the existence of saturated models.

**Theorem 2.2 (Shelah).** *Let  $\lambda$  be a regular cardinal greater than  $\text{LS}(\mathcal{K})$ . If  $\mathcal{K}$  is  $\mu$ -stable for each  $\text{LS}(\mathcal{K}) \leq \mu < \lambda$  then there exists a saturated model of size  $\lambda$ .*

*Proof.* Again we prove this in the case where  $\text{LS}(\mathcal{K}) = \aleph_0$  and  $\lambda = \aleph_1$ . So assume that  $\mathcal{K}$  is  $\aleph_0$ -stable. Construct a strictly  $\prec_{\mathcal{K}}$ -increasing and continuous sequence of countable models  $(M_i : i < \omega_1)$  such that  $M_i \in \mathcal{K}$  and each  $M_{i+1}$  realizes all types in  $S(M_i)$ . This is possible by amalgamation and (A3) using  $\aleph_0$ -stability. Then  $N := \bigcup_{i < \omega_1} M_i \in \mathcal{K}$  has size  $\aleph_1$  and is saturated: Let  $M \prec_{\mathcal{K}} N$  be countable and  $p \in S(M)$  be given. By regularity of  $\omega_1$  there is  $i < \omega_1$  such that  $M \subseteq M_i$  and hence  $M \prec_{\mathcal{K}} M_i$  by (A4). Then  $p$  has an extension in  $S(M_i)$  which is realized in  $M_{i+1}$  by construction, and hence in  $N$ .  $\square$

We give the obvious corollary for categoricity:

**Corollary 2.3 (Shelah).** *Let  $\lambda$  be a regular cardinal greater than  $\text{LS}(\mathcal{K})$ . If  $\mathcal{K}$  is categorical in  $\lambda$  then the model of size  $\lambda$  is saturated.*

*Proof.* This follows immediately from the previous theorem by Theorem 1.4. Notice that we have given a complete proof in the case when  $\lambda = \aleph_1$ .  $\square$

We will now prove that two saturated models of the same size are isomorphic. For this it is easier to use the notion of *monster model* rather than chasing diagrams. We could have introduced the monster model earlier but this is the first point where it provides a real conceptual simplification. We now make precise the notion of universal-homogeneous models.

Let  $\lambda$  be a cardinal. By repeated use of AP and JEP, we can easily construct a model  $N$  which is  $\lambda$ -universal i.e., for each  $M \in \mathcal{K}$  of size less than  $\lambda$  there is a  $\mathcal{K}$ -embedding  $f : M \rightarrow N$ . Moreover, we can also require  $N$  to be  $\lambda$ -model homogeneous i.e., if  $M_1 \prec_{\mathcal{K}} M_2$  of size less than  $\lambda$  and there is a  $\mathcal{K}$ -embedding  $f_1 : M_1 \rightarrow N$  then there exists a  $\mathcal{K}$ -embedding  $f_2 : M_2 \rightarrow N$  extending  $f_1$ . Finally, can further ensure that  $N$  be *strongly  $\lambda$ -model homogeneous* i.e., satisfies in addition that any isomorphism  $f : M_1 \rightarrow M_2$  with  $M_\ell \prec_{\mathcal{K}} N$ , for  $\ell = 1, 2$ , of size less than  $\lambda$  extends to an *automorphism* of  $N$ .

From now until the rest of this chapter, we fix a suitably big cardinal  $\bar{\kappa}$  and a model  $\mathfrak{C}$  which is strongly  $\bar{\kappa}$ -model homogeneous. We will use  $\mathfrak{C}$  as a *monster model*: This is justified since every model of size less than  $\bar{\kappa}$  is isomorphic to a  $\mathcal{K}$ -submodel of  $\mathfrak{C}$  and every type  $p \in S(M)$  with  $M \prec_{\mathcal{K}} \mathfrak{C}$  of size less than  $\bar{\kappa}$  is realized in  $\mathfrak{C}$ .

Furthermore, types over such small  $\mathcal{K}$ -submodels correspond to *orbits* of the automorphism group of  $\mathfrak{C}$  i.e.,  $\text{tp}(a/M, \mathfrak{C}) = \text{tp}(b/M, \mathfrak{C})$  if and only if there exists an automorphism  $F$  of  $\mathfrak{C}$  fixing  $M$  pointwise such that  $F(a) = b$ . We denote by  $\text{Aut}(\mathfrak{C}/M)$  the group of automorphisms of  $\mathfrak{C}$  fixing  $M$  pointwise.

We work inside  $\mathfrak{C}$  and only consider models, sets, and types of size less than  $\bar{\kappa}$ . Since any  $N \prec_{\mathcal{K}} \mathfrak{C}$ , all types  $\text{tp}(a/M, N)$  are of the form  $\text{tp}(a/M, \mathfrak{C})$ , so we will simply write  $\text{tp}(a/M)$ . Recall that by (A4), if  $M, N \prec_{\mathcal{K}} \mathfrak{C}$  are such that  $M \subseteq N$  then  $M \prec_{\mathcal{K}} N$ .

We now introduce a convenient notation that will be used in the proof of the next theorem. Let  $f : M \rightarrow N$  be a  $\mathcal{K}$ -embedding and  $p \in S(M)$  be a type. By  $f(p)$  we mean  $F(p)$ , where  $F$  is any automorphism of  $\mathfrak{C}$  which extends  $f$ . We leave it to the reader to check that this is well-defined.

The monster model point of view allows us to dispense with amalgamation diagrams in favor of more familiar first order monster model arguments but is entirely equivalent. As Baldwin showed in [Ba, Ba1], this simplifies some arguments considerably, in particular the proof of the uniqueness of saturated models below (compare with the proof of the same result in [Gr]).

**Theorem 2.4** (Shelah). (1)  $N$  is  $\mu$ -saturated if and only if  $N$  is  $\mu$ -model homogeneous.  
 (2) Two saturated models  $N_1, N_2$  containing  $M$  such that  $\|N_1\| = \|N_2\| > \|M\|$  are isomorphic over  $M$ .

*Proof.* (2) follows immediately from (1) using a standard back and forth proof so we only prove (1). The right to left direction is trivial and has already been pointed out, so assume that  $N$  is  $\mu$ -saturated. Let  $M \prec_{\mathcal{K}} M' \in \mathcal{K}$  of size  $\lambda < \mu$  and assume that there is a  $\mathcal{K}$ -embedding  $f : M \rightarrow N$ . Write  $M' = M \cup \{a_i : i < \lambda\}$ .

We construct an  $\prec_{\mathcal{K}}$ -increasing and continuous sequence

$$(M_i : i < \lambda)$$

of models in  $\mathcal{K}$  of size  $\lambda$  such that  $M_0 = M$  and  $a_i \in M_{i+1}$ . We do not require  $M_i$  to be contained in  $M'$ . We also construct an increasing and continuous sequence of  $\mathcal{K}$ -embeddings

$$f_i : M_i \rightarrow N$$

such that  $f_0 = f$ .

We do this by induction on  $i < \lambda$ . For  $i = 0$  or a limit, this is easy, so suppose that the  $\mathcal{K}$ -embedding  $f_i : M_i \rightarrow N$  has been constructed. Consider the type  $p_i := \text{tp}(a_i/M_i)$ . Then  $f_i(p_i)$  is a type over  $f_i(M_i) \prec_{\mathcal{K}} N$  of size  $\lambda < \mu$ . Since  $N$  is  $\mu$ -saturated, there is  $b \in N$  realizing  $f_i(p_i)$ . Hence, there exists  $F$  an automorphism of  $\mathfrak{C}$  extending  $f_i$  such that  $F(a_i) = b$ . Choose  $N' \prec_{\mathcal{K}} N$  of size  $\lambda$  containing  $f_i(M_i) \cup b$ . Then  $M_{i+1} := F^{-1}(N') \in \mathcal{K}$  contains  $a_i$  and  $M_i \prec_{\mathcal{K}} M_{i+1}$ . Let  $f_{i+1} := F \upharpoonright M_{i+1} : M_{i+1} \rightarrow N$ . This finishes the construction.

Now we have  $M_\lambda := \bigcup_{i < \lambda} M_i \in \mathcal{K}$  and also  $M_\lambda \prec_{\mathcal{K}} \mathfrak{C}$ . Furthermore,  $f_\lambda := \bigcup_{i < \lambda} f_i : \bigcup_{i < \lambda} M_i \rightarrow N$  is a  $\mathcal{K}$ -embedding. By construction  $M' \subseteq M_\lambda$  and hence  $M' \prec_{\mathcal{K}} M_\lambda$  so  $f_\lambda \upharpoonright M' : M' \rightarrow N$  is a  $\mathcal{K}$ -embedding extending  $f$ .  $\square$

### 3. COUNTABLE SUBSTITUTES TO SATURATED MODELS: UNIVERSAL MODELS, LIMITS, AND SUPERLIMITS

In this chapter we assume that  $(\mathcal{K}, \prec_{\mathcal{K}})$  is an abstract elementary class with amalgamation and arbitrarily large models. We will work inside the monster model  $\mathfrak{C}$  defined in the previous chapter. We assume, in addition, that  $\text{LS}(\mathcal{K}) = \aleph_0$  and that  $\mathcal{K}$  is  $\aleph_0$ -stable.

Notice that both the notion of model homogeneity and the notion of saturation are vacuous unless the models are of size greater than  $\text{LS}(\mathcal{K})$ . If we want to work with models of size  $\text{LS}(\mathcal{K})$  and keep certain uniqueness properties, we need to develop other notions. The first step is to consider universal models *over* a given model (this is also a weakening of model homogeneity). We will then consider limit models, and superlimits. At this level of generality, universal models and limits were introduced by Shelah.

**Definition 3.1.** We say that  $N$  is *universal over*  $M$  if  $M \prec_{\mathcal{K}} N$  and for each  $M'$ , with  $M \prec_{\mathcal{K}} M'$  and  $\|M\| = \|M'\|$ , there is a  $\mathcal{K}$ -embedding  $f : M' \rightarrow N$  which is the identity on  $M$ .

Observe that if  $N$  is universal over  $M$  and  $N \prec_{\mathcal{K}} N'$  then  $N'$  is also universal over  $M$ . By Theorem 2.4, if  $N$  is saturated and  $M \prec_K N$  with  $\|M\| < \|N\|$ , then  $N$  is universal over  $M$ .

The existence of a universal model over  $M$  of size  $\|M\|$  follows from stability in  $\|M\|$ . We are interested here in the countable case. We will iterate the idea of the next proof a number of times, to build limit models from universal ones, and superlimits from limits.

**Proposition 3.2** (Shelah). *For each countable  $M$  there is a countable universal model  $N$  over  $M$ .*

*Proof.* Let  $(M_i : i < \omega)$  be a  $\mathcal{K}$ -increasing sequence of countable models, with  $M_0 = M$ , such that  $M_{i+1}$  realizes every type in  $S(M_i)$ . This is possible by  $\aleph_0$ -stability. Let  $N = \bigcup_{i < \omega} M_i$ . The proof that  $N$  is universal over  $M$  now follows like the similar statement in Theorem 2.4 and is left as an exercise.  $\square$

In general, two countable universal models over  $M$  are not isomorphic. By iterating the idea contained in the previous proof, we obtain limit models: We will show that two limit models over  $M$  are isomorphic.

**Definition 3.3.** Let  $M$  be a countable model. Let  $\alpha < \omega_1$  be a limit ordinal. A countable model  $N$  is an  $\alpha$ -*limit model over*  $M$  if there exists an  $\prec_{\mathcal{K}}$ -increasing and continuous chain  $(N_i : i < \alpha)$  such that

- (1)  $N_0 = M$ ,

- (2)  $N_{i+1}$  is universal over  $M_i$ ,
- (3)  $N = \bigcup_{i < \alpha} N_i$ .

We say that  $(N_i : i < \alpha)$  is an  $\alpha$ -tower for  $N$  over  $M$ .

It is immediate that if  $N$  is an  $\alpha$ -limit over  $M$  then  $N$  is universal over  $M$  since it extends  $N_1$  which is universal over  $M$  (in the notation of the previous definition).

Observe also that if  $N$  is an  $\alpha$ -limit over  $M$ , then  $N$  is an  $\omega$ -limit over  $M$ : If  $(N_i : i < \alpha)$  is an  $\alpha$ -tower for  $N$  over  $M$ , choose  $(\alpha_n : n < \omega)$  a cofinal sequence for  $\alpha$  with  $\alpha = 0$ . Then  $(N_{\alpha_n} : n < \omega)$  is an  $\omega$ -tower for  $N$  over  $M$  as  $N_{\alpha_{n+1}}$  is universal over  $N_{\alpha_n}$ . Observe finally that for any countable  $M$  and any limit ordinal  $\alpha < \omega_1$ , there exists an  $\alpha$ -limit  $N$  over  $M$  by repeated applications of Proposition 3.2.

We now prove two facts about limit models, which are adapted from Shelah's Lemma 2.2 in [Sh394]; they are stated and proved in [Sh88]. Analogous arguments for uncountable cardinalities are much more difficult (see [VD]).

**Proposition 3.4.** *Let  $\alpha_1, \alpha_2 < \omega_1$  be limit ordinals. Let  $M$  be countable and assume that  $N_\ell$  is an  $\alpha_\ell$ -limit over  $M$ . Then  $N_1 \cong_M N_2$ .*

*Proof.* Without loss of generality, we may assume that  $\alpha_1 = \omega = \alpha_2$ . Let  $(N_n^\ell : n < \omega)$  be an  $\omega$ -tower for  $N^\ell$  over  $M$ , for  $\ell = 1, 2$ . Proving the isomorphism between  $N_1$  and  $N_2$  is now a standard back-and-forth construction using the universality of  $N_{n+1}^\ell$  over  $N_n^\ell$ : We construct an increasing sequence of  $\mathcal{K}$ -embeddings  $f_n$  such that  $\text{dom}(f_{2n})$  contains  $N_n^1$  and  $\text{im}(f_{2n+1})$  contains  $N_n^2$ , with  $f_0 = \text{id}_M$ . This is possible, since each  $N_{n+1}^\ell$  is universal over  $N_n^\ell$ , for  $\ell = 1, 2$ . It is enough, as the union of the  $f_n$ s is an isomorphism between  $N_1$  and  $N_2$  which is the identity on  $M$ .  $\square$

Since the value of the ordinal  $\alpha$  is immaterial, we will simply say that  $N$  is a *limit* over  $M$ , when  $N$  is an  $\alpha$ -limit over  $M$ . The next proposition is proved by pasting the towers witnessing the limits together.

**Proposition 3.5.** *Let  $\alpha < \omega_1$  be an ordinal, not necessarily a limit. Assume that  $(N_i : i < \alpha)$  is  $\prec_{\mathcal{K}}$ -increasing and continuous such that  $N_{i+1}$  is a limit over  $N_i$ , for  $i < \alpha$ . Then  $\bigcup_{i < \alpha} N_i$  is a limit over  $N_0$ .*

We now consider superlimits, which are obtained from limits the way limits were obtained from universal models.

**Definition 3.6.** Let  $\alpha < \omega_1$  be a limit ordinal. Let  $M$  be a countable model. A countable model  $N$  is an  $\alpha$ -*superlimit* over  $M$  if there exists an  $\prec_{\mathcal{K}}$ -increasing and continuous chain  $(N_i : i < \alpha)$  such that

- (1)  $N_0 = M$ ,
- (2)  $N_{i+1}$  is a limit over  $N_i$ ,
- (3)  $\bigcup_{i < \alpha} N_i = N$ .

We call  $(N_i : i < \alpha)$  as above an  $\alpha$ -supertower for  $N$  over  $M$ .

Superlimits exist by  $\aleph_0$ -stability. Clearly, a superlimit is a limit, since if  $M_{i+1}$  is a limit over  $M_i$  then  $M_{i+1}$  is universal over  $M_i$ . An  $\alpha$ -superlimit is also an  $(\omega \cdot \alpha)$ -limit, by unravelling the definitions. So, we clearly have the uniqueness property, but we also have a stronger one.

The next proposition shows that it is enough to understand  $\omega$ -superlimits.

**Proposition 3.7.** *Let  $\alpha < \omega_1$  be a limit ordinal. Let  $M$  be countable. Every  $\alpha$ -superlimit over  $M$  is an  $\omega$ -superlimit over  $M$ .*

*Proof.* As  $\alpha$  is a countable limit ordinal, there is  $(\alpha_n : n < \omega)$  cofinal in  $\alpha$ , with  $\alpha_0 = 0$ . Then if  $(N_i : i < \alpha)$  is an  $\alpha$ -supertower for  $N$  over  $M$ , then  $(N_{\alpha_n} : n < \omega)$  is clearly an  $\omega$ -supertower for  $N$  over  $M$ , as  $N_{\alpha_{n+1}}$  is a limit over  $N_{\alpha_n}$  by Proposition 3.5.  $\square$

So we focus on  $\omega$ -superlimits. The proof of the next proposition is simply an iteration of Proposition 3.4.

**Proposition 3.8.** *Let  $N, N'$  be  $\omega$ -superlimits over  $M$ . Then  $N \cong_M N'$ . Furthermore, if  $(N_n : n < \omega)$  and  $(N'_n : n < \omega)$  are  $\omega$ -supertowers for  $N$  and  $N'$  (respectively) over  $M$ , then there exists an isomorphism  $f : N \cong_M N'$  such that*

$$f \upharpoonright N_n : N_n \cong N'_n, \quad \text{for each } n < \omega.$$

We will use the following convenient piece of notation: Given  $N$  an  $\alpha$ -superlimit over  $M$ , we denote by  $\bar{N}$  some  $\alpha$ -supertower  $(N_i : i < \alpha)$  for  $N$  over  $M$ . We also write  $f : \bar{N} \cong \bar{N}'$  for an isomorphism between the two supertowers of two superlimits  $N$  and  $N'$  as in the previous proposition.

We will show that countable unions of  $\omega$ -superlimits are  $\omega$ -superlimits under the right circumstances. We will need to consider sequences of supertowers  $\bar{N}_i$ , so it is natural to order them. The most natural choice is to consider the following partial order  $\leq$  between  $\omega$ -supertowers:

$$(N_n : n < \omega) \leq (N'_n : n < \omega),$$

if for each  $n < \omega$ ,  $N'_n$  is a limit over  $N_n$ . Unfortunately, this is too strong for our purposes, so we consider the weakening  $\leq^*$ , where the  $*$  serves, as usual, to denote eventual domination: We say that

$$(N_n : n < \omega) \leq^* (N'_n : n < \omega),$$

if for each  $n < \omega$ , there exists  $m \geq n$  such that  $N'_m$  is a limit over  $N_n$ . Notice that  $N'_k$  is a limit over  $N_n$  for each  $k \geq m$  by Proposition 3.5. It is clear that  $\leq^*$  is transitive, and if  $\bar{N} \leq \bar{N}'$  then  $\bar{N} \leq^* \bar{N}'$ .

We now show that unions of superlimits produce superlimits, provided the supertowers are  $\leq^*$ -ordered. The notation is a bit cumbersome, but the proof is a straightforward diagonal argument.

**Proposition 3.9.** *Let  $\alpha < \omega_1$  be a limit ordinal. Let  $(N^i : i < \alpha)$  be an  $\prec_{\mathcal{K}}$ -increasing and continuous chain of  $\omega$ -superlimits over  $M$  with  $\omega$ -supertowers  $\bar{N}^i$ , for  $i < \alpha$ . Suppose, in addition, that*

$$\bar{N}^i \leq^* \bar{N}^j, \quad \text{for } i < j < \alpha.$$

*Then  $\bigcup_{i < \alpha} N^i$  is an  $\omega$ -superlimit over  $M$ . Moreover, there exists  $(N_n^\alpha : n < \omega)$  an  $\omega$ -supertower for  $\bigcup_{i < \alpha} N^i$  over  $M$  such that*

$$\bar{N}^i \leq^* (N_n^\alpha : n < \omega), \quad \text{for each } i < \alpha.$$

*Proof.* It is enough to prove the last sentence. Also, by choosing a cofinal sequence  $(\alpha_n : n < \omega)$  for  $\alpha$  with  $\alpha_0 = 0$ , and using the transitivity of  $\leq^*$ , we may assume that  $\alpha = \omega$ . So we consider an  $\leq^*$ -increasing sequence  $(\bar{N}^i : i < \omega)$  of  $\omega$ -supertowers  $\bar{N}^i$  for  $N^i$  over  $M$ . We will construct a strictly increasing function  $f : \omega \rightarrow \omega$  such that  $f(0) = 0$  and for each integer  $n \geq 0$

$$N_{f(n+1)}^{n+1} \text{ is a limit over } N_{f(n)}^n \text{ and over } N_k^i, \quad \text{for each } i, k \leq n.$$

This is enough: Let  $N_n^\omega := N_{f(n)}^n$ , for each  $n < \omega$ . Then  $(N_n^\omega : n < \omega)$  is an  $\omega$ -supertower for  $\bigcup_{i < \omega} N^i$  over  $M$  since  $N_0^\omega = M$  and  $N_{n+1}^\omega$  is a limit over  $N_n^\omega$  by the definitions. Furthermore,

$$\bar{N}^i \leq^* (N_n^\omega : n < \omega), \quad \text{for each } i < \omega :$$

Let  $i, k < \omega$  be given and consider  $n := \max(i, k)$ . Then  $N_{n+1}^\omega (= N_{f(n+1)}^{n+1})$  is a limit over  $N_k^i$  by definition.

It remains to show that such an  $f$  can be found. By definition of  $\leq^*$ , for each  $i < \omega$  there exists a strictly increasing function  $f_i : \omega \rightarrow \omega$  such that

$$N_{f_i(n)}^{i+1} \text{ is a limit over } N_n^i, \quad \text{for each } n < \omega.$$

We define  $f(n)$  by induction on  $n$ . Let  $f(0) = 0$ . Having constructed  $f(n)$ , we define  $f(n+1)$  by taking the maximum of the following three numbers:

$$f_n(f(n)), \quad f_n \circ f_{n-1} \circ \cdots \circ f_0(n), \quad f(n) + 1.$$

Then  $f(n+1)$  is as required: The fact that  $f(n+1)$  is at least the first number ensures that  $N_{f(n+1)}^{n+1}$  is a limit over  $N_{f(n)}^n$  and also over  $N_k^i$  for  $i \leq n$  and  $k < n$ .

That  $f(n + 1)$  is at least the second number ensures that  $N_{f(n+1)}^{n+1}$  is a limit over each  $N_n^i$ , for  $i \leq n$ , since

$$f_n \circ f_{n-1} \circ \cdots \circ f_0(n) \geq f_n \circ f_{n-1} \circ \cdots \circ f_i(n).$$

And finally,  $f(n + 1) > f(n)$  since  $f(n + 1)$  is at least the third number. This finishes the proof.  $\square$

The last two propositions will be used to show that categoricity in  $\aleph_1$  implies the absence of Vaughtian pairs (Theorem 6.5).

## 4. THE PROBLEM OF TYPES: TAMENESS AND LOCALITY

In this chapter we assume that  $(\mathcal{K}, \prec_{\mathcal{K}})$  is an abstract elementary class with amalgamation and arbitrarily large models. We work inside the monster model  $\mathfrak{C}$  and return to the problem of types.

Let  $p \in S(M)$  and  $q \in S(N)$ , with  $M \prec_{\mathcal{K}} N$ . We say that  $q$  *extends*  $p$  if some (equivalently any) realization of  $q$  realizes  $p$ . We will write  $p \subseteq q$  if  $q$  extends  $p$ , in spite of the fact that types are not sets of formulas.

Consider an  $\subseteq$ -increasing chain of types  $(p_i : i < \delta)$ , say with  $p_i \in S(M_i)$ . The first question is *existence*: Is there  $a \in \mathfrak{C}$  such that  $a$  realizes  $p_i$ , for each  $i < \delta$  (unions of types are really intersections of orbits). The second question is *uniqueness*: If  $a, b \in \mathfrak{C}$  realize  $p_i$  for each  $i < \delta$  and  $M_\delta = \bigcup_{i < \delta} M_i$  do we necessarily have

$$\text{tp}(a/M_\delta) = \text{tp}(b/M_\delta)?$$

The answer to both questions is *no* in general; concrete examples are provided in [BaSh]. In order to deal with the first question, we introduce the following definition, which appears implicitly in [Sh394].

**Definition 4.1.** An  $\subseteq$ -increasing chain of Galois types  $(p_i : i < \delta)$  with  $p_i \in S(M_i)$  is *coherent* if there exist elements  $a_i \in \mathfrak{C}$  and  $F_{i,j} \in \text{Aut}(\mathfrak{C}/M_i)$ , for  $i < j < \delta$ , such that:

- (1)  $p_i = \text{tp}(a_i/M_i)$ ;
- (2)  $F_{i,j}(a_j) = a_i$  for  $i < j < \delta$ .
- (3)  $F_{i,j} = F_{i,k} \circ F_{k,j}$  for any  $i < k < j < \delta$ .

The next proposition implies that the union of a coherent chain of Galois types is realized.

**Proposition 4.2.** *Let  $(p_i : i < \delta)$  be a coherent chain of types, with  $p_i \in S(M_i)$ . Then there exists  $p_\delta \in S(M_\delta)$ , with  $M_\delta = \bigcup_{i < \delta} M_i$ , such that  $(p_i : i < \delta + 1)$  is a coherent chain of types.*

*Proof.* Let  $a_i \models p_i$  and  $F_{i,j} \in \text{Aut}(\mathfrak{C}/M_i)$ , for  $i < j < \delta$ , witness the coherence of  $(p_i : i < \delta)$ . Let  $M_\delta = \bigcup_{i < \delta} M_i$ . We need to find  $a_\delta$  so that for  $p_\delta = \text{tp}(a_\delta/M_\delta)$  there are  $F_{i,\delta}$  for  $i < \delta$  demonstrating that  $(p_i : i < \delta + 1)$  is coherent.

Let  $g_i = F_{0,i} \upharpoonright M_i : M_i \rightarrow \mathfrak{C}$ . Notice that the sequence  $(g_i : i < \delta)$  of  $\mathcal{K}$ -embeddings is increasing and continuous. Hence we can find  $G \in \text{Aut}(\mathfrak{C})$  extending  $\bigcup_{i < \delta} g_i$ . Let  $a_\delta = G^{-1}(a_0)$  and define  $F_{i,\delta} = F_{0,i}^{-1} \circ G$ . Then  $F_{i,\delta}$  fixes  $M_i$  since  $G$  extends  $F_{0,i} \upharpoonright M_i$  and sends  $a_\delta$  to  $a_i$ . Furthermore,  $F_{i,j} \circ F_{j,\delta} = F_{i,j} \circ F_{0,j}^{-1} \circ G = F_{0,i} \circ G = F_{i,\delta}$ .  $\square$

**Remark 4.3.** Since any  $\subseteq$ -increasing chain of Galois types  $(p_i : i < \omega)$  is coherent, the previous proposition shows that its union is realized. Since any countable ordinal is either a successor or has cofinality  $\omega$ , we derive easily from this that the union of any countable chain of types is realized. Without further assumptions, this may fail for longer chains in general.

We first consider tameness.

**Definition 4.4.** Let  $\chi$  be an infinite cardinal. We say that  $\mathcal{K}$  is  $\chi$ -tame, if whenever  $p \neq q \in S(N)$ , there exists  $M \prec_{\mathcal{K}} N$  of size  $\chi$  such that  $p \upharpoonright M \neq q \upharpoonright M$ . We will say that  $\mathcal{K}$  is *tame* if  $\mathcal{K}$  is  $\text{LS}(\mathcal{K})$ -tame.

- Remark 4.5.**
- (1) If  $\mathcal{K}$  is first order, homogeneous,  $L^n$  with amalgamation or excellent then  $\mathcal{K}$  is  $\chi$ -tame for  $\chi = \text{LS}(\mathcal{K})$ .
  - (2) Shelah introduced tameness in [Sh394]. He refers to this issue as the main difficulty. It was used as a separate assumption for the first time in [GrVD].
  - (3) It follows from Shelah's result in [Sh394] that if  $\mathcal{K}$  is categorical in arbitrarily large cardinals, then  $\mathcal{K}$  is  $\chi$ -tame for some  $\chi$  less than the Hanf number.
  - (4) It is not clear at this stage, how strong tameness is. In the interesting particular cases considered by Zilber and Gavrilovich, tameness is established by proving 'excellence,' though sometimes only an excellence-like condition, as the context is not strictly  $L_{\omega_1, \omega}$ . The advantage of excellence is that it is a condition involving only countable models, whereas tameness involves uncountable models also. The disadvantage is that it is far more complicated. Also, it follows from the upward categoricity theorem in [Le2] and Shelah's results on categorical sentences in  $L_{\omega_1, \omega}$  [Sh48] that it is consistent with ZFC that any tame  $L_{\omega_1, \omega}$ -class with AP and arbitrarily large models that is categorical in  $\aleph_1$  is excellent.

And now locality:

**Definition 4.6.** We say that  $\mathcal{K}$  is *local* if whenever  $p \neq q \in S(N)$  and  $N = \bigcup_{i < \mu} N_i$ , for  $\mu$  a cardinal, then there is  $i < \mu$  such that  $p \upharpoonright N_i \neq q \upharpoonright N_i$ .

Notice that if  $\mathcal{K}$  is first order, homogeneous, or excellent, then  $\mathcal{K}$  is local.

**Remark 4.7.** In [HyVi], Hyttinen and Viljanen introduce the notion of *finitary* AEC (initially called 'local'): A class  $\mathcal{K}$  with amalgamation is *finitary* if  $M \prec_{\mathcal{K}} N$  if and only if  $M \subseteq N$  and  $\text{tp}(a/\emptyset, M) = \text{tp}(a/\emptyset, N)$  for each finite  $a \in M$ . (The definition of Galois types over the empty set is done exactly as in the first chapter.) All the examples of abstract elementary classes we have given are finitary. They prove that if an AEC is finitary and  $M$  is countable, then if  $\text{tp}(a/M) \neq \text{tp}(b/M)$  there exists a finite  $A \subseteq M$  such that  $a$  and  $b$  are not automorphic over  $A$ . In conjunction with tameness, finitary then implies that the same holds for all models which implies locality.

Locality implies tameness:

**Proposition 4.8.** *If  $\mathcal{K}$  is local then  $\mathcal{K}$  is tame.*

*Proof.* We prove by induction on  $\mu$  that if  $\text{tp}(a/M) \neq \text{tp}(b/M)$ , for  $M$  of size  $\mu$ , then there is  $M' \prec_{\mathcal{K}} M$  of size  $\text{LS}(\mathcal{K})$  such that  $\text{tp}(a/M') \neq \text{tp}(b/M')$ .

For  $\mu = \text{LS}(\mathcal{K})$ , there is nothing to show. Now assume that  $\mu > \text{LS}(\mathcal{K})$ . Let  $M$  be given. Choose  $(M_i : i < \mu)$  increasing and continuous such that  $\|M_i\| = |i| + \aleph_0$  and  $\bigcup_{i < \mu} M_i = M$ . If  $\text{tp}(a/M) \neq \text{tp}(b/M)$ , then there is  $i < \mu$  such that  $\text{tp}(a/M_i) \neq \text{tp}(b/M_i)$ , since  $\mathcal{K}$  is local. But by induction hypothesis, there is  $M' \prec_{\mathcal{K}} M_i$  of size  $\text{LS}(\mathcal{K})$  such that  $\text{tp}(a/M') \neq \text{tp}(b/M')$ . So, we are done since  $M' \prec_{\mathcal{K}} N$ .  $\square$

We now show that the answer to both existence and uniqueness question is positive when  $\mathcal{K}$  is local:

**Proposition 4.9.** *Assume that  $\mathcal{K}$  is local. Let  $(M_i : i \leq \delta)$  be an increasing and continuous sequence of models, and  $(p_i \in S(M_i) : i < \delta)$  be an  $\subseteq$ -increasing sequence of types. Then there is a unique  $p \in S(M_\delta)$  extending each  $i < \delta$ .*

*Proof.* Uniqueness follows easily: If  $\delta$  is a successor, there is nothing to show, so we may assume that  $\delta$  is a limit. By taking a cofinal subsequence if necessary, we may assume that  $\delta$  is a cardinal, so uniqueness follows immediately from the fact that  $\mathcal{K}$  is local.

For existence, assume that  $(p_i : i < \delta)$  is given. We show by induction on  $i < \delta$  that  $(p_j : j \leq i)$  is coherent. For  $i = 0$  or a successor, this is easy. Assume that  $i$  is a limit and that  $(p_j : j < i)$  is coherent. Then by Proposition 4.2, there exists  $p'_i \in S(M_i)$  such that  $(p_j, p'_i : j < i)$  is coherent. But  $p'_i \upharpoonright M_j = p_i \upharpoonright M_j$ , for each  $j < i$  by definition. Hence by uniqueness, we must have  $p'_i = p_i$ , which shows that  $(p_j : j \leq i)$  is coherent. Thus  $(p_i : i < \delta)$  is coherent, and so there exists  $p \in S(M_\delta)$  extending each  $p_i$  by another application of Proposition 4.2.  $\square$

## 5. BIG AND QUASIMINIMAL TYPES

In this chapter we assume that  $(\mathcal{K}, \prec_{\mathcal{K}})$  is an abstract elementary class with amalgamation and arbitrarily large models. We work inside  $\mathfrak{C}$ . We assume, in addition, that  $\text{LS}(\mathcal{K}) = \aleph_0$  and that  $\mathcal{K}$  is  $\aleph_0$ -stable.

We will introduce big and quasiminimal types. They are the natural counterpart of nonalgebraic and strongly minimal types at this level of generality.

We say that a type  $p \in S(M)$  is *nonalgebraic* if  $p$  is realized outside of  $M$ . With amalgamation, any type has an extension, but a non-algebraic type may have a bounded number of solutions and thus no nonalgebraic extension to a model that contains all of them. The next definition is a strengthening of nonalgebraicity to avoid these types.

**Definition 5.1.** Let  $p \in S(M)$  with  $M$  countable. We say that  $p$  is *big* if  $p$  has a nonalgebraic extension to any countable  $M'$  with  $M \prec_{\mathcal{K}} M'$ .

Notice that if  $p \in S(N)$  is big and  $M \prec_{\mathcal{K}} N$  then  $p \upharpoonright M$  is big. The next proposition will allow us to find big types.

**Proposition 5.2.** Let  $p \in S(M)$  and  $M$  countable. The following conditions are equivalent:

- (1)  $p$  is big;
- (2)  $p$  has a nonalgebraic extension to some  $M'$  universal over  $M$ ;
- (3)  $p$  is realized uncountably many times in  $\mathfrak{C}$ .

*Proof.* (1) implies (2) by definition, since there exists a countable universal model  $M'$  over  $M$  by Proposition 3.2. (2) implies (3): Let  $M'$  be a universal model over  $M$  and let  $p' \in S(M')$  be a nonalgebraic extension of  $p$ . Suppose that  $A \subseteq \mathfrak{C}$  is a countable set of realizations of  $p$ . Let  $N$  be countable containing  $A \cup M$ . By universality of  $M'$  over  $M$ , we may assume that  $N \prec_{\mathcal{K}} M'$ . Since  $p$  is realized outside  $N$  (any realization of  $p'$ ), then  $A$  does not contain all the realizations of  $p$  in  $\mathfrak{C}$ , so  $p$  must be realized uncountably many times in  $\mathfrak{C}$ . Finally (3) implies (1) is clear, as  $p$  must be realized outside any countable model containing  $M$ .  $\square$

We now show that big types exist, and not only do they have nonalgebraic extensions, but they also have big extensions.

**Proposition 5.3.** There exists a big type  $p \in S(M)$ , for each countable  $M$ . Moreover, if  $p \in S(M)$  is big and  $M'$  is countable containing  $M$ , then there is a big  $p' \in S(M')$  extending  $p$ .

*Proof.* Let  $M$  be given. Choose  $N$  countable universal over  $M$ . Then any nonalgebraic  $q \in S(N)$  is such that  $q \upharpoonright M \in S(M)$  is big by the previous proposition.

Moreover, if  $p \in S(M)$  is big and  $M'$  is countable containing  $M$ , we can choose  $N$  countable universal over  $M'$ . Since  $p$  is big,  $p$  has a nonalgebraic extension  $q \in S(N)$ ; again  $q \upharpoonright M'$  is big by the previous proposition.  $\square$

We finally consider the simplest big types. Observe that we only ask about extensions on models of the same size.

**Definition 5.4.** Let  $p \in S(M)$  with  $M$  countable. We say that  $p$  is *quasiminimal* if  $p$  is big and has exactly one big extension in  $S(M')$  for any countable  $M'$  with  $M \prec_{\mathcal{K}} M'$ .

The name quasiminimal is consistent with Zilber's usage, since each quasiminimal type is realized uncountably many times but has at most one extension which is realized uncountably many times. We can now show that quasiminimal types exist by using the usual tree argument:

**Theorem 5.5.** *There exists a quasiminimal type over some countable model. Moreover, if  $p \in S(M)$  is big and  $M$  is countable, then there is a countable  $M'$  extending  $M$  and a quasiminimal  $p' \in S(M')$  extending  $p$ .*

*Proof.* Since big types exist by the previous proposition, it is enough to show the second sentence. Let  $p \in S(M)$  be big and suppose, for a contradiction, that  $p$  has no quasiminimal extension over a countable model. Since  $p$  has a big extension over any model by the previous proposition, this means that each big extension of  $p$  has at least two big extensions over some model. We can therefore construct a tree of types  $(p_\eta : \eta \in {}^{\omega}2)$  with  $p_\eta \in S(M_\eta)$  and  $M_\eta$  countable, such that

- (1)  $M_\emptyset = M$  and  $p_\emptyset = p$ ;
- (2)  $(p_{\eta \upharpoonright n} : n < \ell(\eta))$  is  $\subseteq$ -increasing;
- (3)  $p_\eta$  is big;
- (4)  $M_{\eta \uparrow 0} = M_{\eta \uparrow 1}$  but  $p_{\eta \uparrow 0} \neq p_{\eta \uparrow 1}$ .

But this contradicts  $\aleph_0$ -stability: Let  $\eta \in {}^{\omega}2$ . Since  $(p_{\eta \upharpoonright n} : n < \omega)$  is countable and increasing, there is  $p_\eta$  extending each  $p_{\eta \upharpoonright n}$  by Remark 4.3. Let  $N$  be countable containing  $\bigcup_{\eta \in {}^{\omega}2} M_\eta$ . Each  $p_\eta$  for  $\eta \in {}^{\omega}2$  has an extension in  $S(N)$ , so there are  $2^{\aleph_0}$  types over  $N$ , a contradiction.  $\square$

## 6. VAUGHTIAN PAIRS

In this chapter we assume that  $\mathcal{K}$  is a local abstract elementary class, with amalgamation and arbitrarily large models. We work inside  $\mathfrak{C}$ . We assume that  $\text{LS}(\mathcal{K}) = \aleph_0$  and that  $\mathcal{K}$  is  $\aleph_0$ -stable.

We start with a result on existence and uniqueness of nonalgebraic extensions over certain models: countable limit models and over saturated models of cardinality  $\aleph_1$ . Locality is used here for the first time.

**Proposition 6.1.** *Suppose  $\mathcal{K}$  is local. Let  $p \in S(M)$  be quasiminimal, with  $M$  countable, and let  $N$  be a limit over  $M$ . Then there is a unique nonalgebraic extension of  $p$  in  $S(N)$ .*

*Proof.* Let  $(N_n : n < \omega)$  be an  $\omega$ -tower for  $N$  over  $M$ . Let  $q \in S(N)$  be the unique big type extending  $p$  in  $S(M)$ . Then  $q$  is nonalgebraic, which proves existence. Now assume that  $q' \in S(N)$  is a nonalgebraic extension of  $p$ . Let  $n < \omega$ . Then  $q \upharpoonright N_n$  and  $q' \upharpoonright N_n$  are two big extensions of  $p$ ; the first by restriction, and the second by Proposition 5.2 since  $q' \upharpoonright N_{n+1}$  is nonalgebraic extension of  $q' \upharpoonright N_n$  and  $N_{n+1}$  is universal over  $N_n$ . Hence, by quasiminimality of  $p$ , we have  $q \upharpoonright N_n = q' \upharpoonright N_n$ . Since this holds for any  $n < \omega$ , we have that  $q = q'$  by locality of  $\mathcal{K}$ .  $\square$

We can extend the previous result to the saturated model of size  $\aleph_1$  (which exists by Theorem 2.2):

**Proposition 6.2.** *Suppose  $\mathcal{K}$  is local. Let  $p \in S(M)$  be quasiminimal, with  $M$  countable. There is a unique nonalgebraic extension of  $p$  to any saturated model  $N$  of size  $\aleph_1$  containing  $M$ .*

*Proof.* First, there can be at most one nonalgebraic extension of  $p$  over the model saturated model of size  $\aleph_1$ , since it is saturated: If  $q_1 \neq q_2 \in S(N)$ , with  $N$  of size  $\aleph_1$  both extend  $p$ , then, since locality implies tameness, there is  $M'$  countable, with  $M \prec_{\mathcal{K}} M' \prec_{\mathcal{K}} N$ , such that  $q_1 \upharpoonright M' \neq q_2 \upharpoonright M'$ . Since  $N$  is saturated, it is universal over  $M'$ , and hence  $q_1 \upharpoonright M'$  and  $q_2 \upharpoonright M'$  are big, by Proposition 5.2. This contradicts the quasiminimality of  $p$ .

We now turn to existence and notice that by Theorem 2.4, it is enough to prove it for some saturated model of size  $\aleph_1$ . Consider an increasing and continuous chain of models  $(M_i : i < \aleph_1)$ , such that  $M_0 = M$ , and  $M_{i+1}$  is a limit over  $M_i$ . This is possible by  $\aleph_0$ -stability. Notice that each  $M_{i+1}$  realizes every type over  $M_i$  by universality. Hence the model  $N = \bigcup_{i < \aleph_1} M_i$  is saturated, and by Proposition 3.5, each  $M_i$  is a limit over  $M$ . Let  $p_0 = p$ . By Proposition 6.1, there is a unique nonalgebraic  $p_i \in S(M_i)$  extending  $p_0$ . By uniqueness, the sequence

$(p_i : i < \aleph_1)$  is  $\subseteq$ -increasing, and so there is  $q \in S(N)$  extending each  $i < \omega_1$  by Proposition 4.9. Then,  $q$  is clearly nonalgebraic, as each  $p_i$  is.  $\square$

We now define Vaughtian pairs with respect to quasiminimal types (they were introduced with respect to minimal types by Shelah in [Sh394]):

**Definition 6.3.** Let  $p \in S(M)$  be quasiminimal, with  $M$  countable. A  $(p, \mu)$ -Vaughtian pair is a pair of models  $N_1, N_2$  of size  $\mu$  with  $M \prec_{\mathcal{K}} N_1 \prec_{\mathcal{K}} N_2$ ,  $N_1 \neq N_2$ , such that there is a nonalgebraic extension of  $p$  in  $S(N_1)$  which is not realized in  $N_2$ .

Let  $p \in S(M)$  be quasiminimal with  $M$  countable. The goal is to prove that there are no  $(p, \mu)$ -Vaughtian pairs for any uncountable  $\mu$ . In order to extend the traditional Vaught argument, we need to use a countable substitute for the notion  $N$  is *saturated over  $M$* . In the excellent case, one can use countable *full* models over  $M$ : Two countable full models over  $M$  are isomorphic over  $M$  and the union of any countable chain of full models over  $M$  is full over  $M$ . The key here is to use superlimits, which have good uniqueness properties (Proposition 3.8), and behave well under unions (Proposition 3.9).

The next easy observation is used in the proof that there are no Vaughtian pairs:

**Proposition 6.4.** *Suppose that  $M_0, M_1$  are countable and  $A$  is a countable set. There exists a countable  $N$  containing  $M_0 \cup M_1 \cup A$  which is a limit over both  $M_0$  and  $M_1$ .*

*Proof.* It is enough to find a countable model  $N$  which is universal over  $M_0$  and  $M_1$  and contains  $A$ . But this is clear: Choose first  $N'$  containing  $A$  which is universal over  $M_0$ . Now choose  $N''$  containing  $N' \cup M_1$  which is universal over  $M_1$ . Since  $N' \prec_{\mathcal{K}} N''$  and  $N'$  is universal over  $M_0$ , then so is  $N''$ .  $\square$

We now prove that there are no Vaughtian pairs. This is the main use of superlimits; this allows us to circumvent the fact that we do not have categoricity in two successive cardinals.

**Theorem 6.5.** *Assume that  $\mathcal{K}$  is categorical in  $\aleph_1$ . Let  $p \in S(M)$  be quasiminimal with  $M$  countable. Then there are no  $(p, \mu)$ -Vaughtian pairs, with  $\mu \geq \aleph_1$ .*

*Proof.* Suppose that  $N_0 \prec_{\mathcal{K}} N_1$  is a  $(p, \mu)$ -Vaughtian pair, for  $\mu \geq \aleph_1$ . By the usual  $\omega$ -chain argument, we may assume that  $\mu = \aleph_1$ , and hence that  $N_0$  and  $N_1$  are saturated by Corollary 2.3.

We now construct a  $(p, \aleph_0)$ -Vaughtian pair  $N^0 \prec_{\mathcal{K}} N^1$  such that  $N^\ell$  is an  $\omega$ -superlimit over  $M$ , with  $\omega$ -supertower  $\bar{N}^\ell$ , for  $\ell = 0, 1$ , and such that

$$\bar{N}^0 \leq^* \bar{N}^1.$$

Let  $N_0^\ell = M$  for  $\ell = 0, 1$ . Choose a limit  $N_1^0$  over  $M$  such that  $N_1^0 \prec_{\mathcal{K}} N_0$  (this is possible since  $N_0$  is saturated). Now choose  $N_1^1 \prec_{\mathcal{K}} N_1^0$  a limit over  $N_1^0$  containing an element  $a \in N_1^1 \setminus N_0$ . Now having constructed  $N_n^0 \prec_{\mathcal{K}} N_0$  and  $N_n^1 \prec_{\mathcal{K}} N_1^0$  countable with  $N_n^1$  a limit over  $N_n^0$ , choose  $N_{n+1}^0 \prec_{\mathcal{K}} N_0$  a limit over  $N_n^0$  containing all the realizations of the unique big extension of  $p$  to  $N_n^0$  in  $N_n^1$  (this is possible since this set is countable). Now choose countable  $N_{n+1}^1 \prec_{\mathcal{K}} N_1^0$  a limit over both  $N_n^1$  and  $N_{n+1}^0$  (this is possible by Proposition 6.4,  $N_{n+1}^1$  can be chosen inside  $N_1^1$  by the saturation of  $N_1^1$ ). Let  $N^\ell = \bigcup_{n < \omega} N_n^\ell$ , for  $\ell = 0, 1$ . Then  $N^\ell$  is an  $\omega$ -superlimit over  $M$  with  $\omega$ -supertower  $\bar{N}^\ell$  such that  $\bar{N}^0 \leq^* \bar{N}^1$  (even  $\bar{N}^0 \leq \bar{N}^1$ ). Furthermore,  $N^0 \prec_{\mathcal{K}} N^1$  forms a  $(p, \aleph_0)$ -Vaughtian pair. Let  $p_0$  be the unique big type extending  $p$  in  $S(N^0)$ , which exists by countability of  $N^0$  and quasiminimality of  $p$ .

To contradict categoricity in  $\aleph_1$ , we construct an increasing and continuous chain  $(N^i : i < \aleph_1)$  of  $\omega$ -superlimits over  $M$ , such that  $N^i \neq N^{i+1}$  with a big extension  $p_i \in S(N^i)$  of  $p$  which is not realized in  $N^{i+1}$ , and such that the sequence of limits  $(\bar{N}^i : i < \omega_1)$  is  $\leq^*$ -increasing: We do this by induction on  $i < \omega_1$ . For  $i = 0$ , this is given. At limit  $i < \omega_1$ , let  $N^i = \bigcup_{j < i} N^j$  with  $\omega$ -supertower  $\bar{N}^i$  over  $M$  as in Proposition 3.9. Now having constructed the  $\omega$ -superlimit model  $N^i$  with  $\omega$ -supertower  $\bar{N}^i$  over  $M$ , for  $i$  limit or successor, choose an isomorphism  $f_i : \bar{N}^0 \cong \bar{N}^i$  as in Proposition 3.8. Then  $f_i$  extends to an automorphism  $F \in \text{Aut}(\mathcal{E}/M)$  and we let  $p_i = F(p_0)$ ,  $N^{i+1} = F(N^1)$ , and  $\bar{N}^{i+1} = F(\bar{N}^1)$ . Then  $p_i \in S(N^i)$  is a big extension of  $p$  which is not realized in  $N^{i+1}$ , and  $\bar{N}^i \leq^* \bar{N}^{i+1}$ , since  $F$  is an automorphism extending  $f_i$  (and hence respecting levels) and  $\bar{N}^0 \leq^* \bar{N}^1$ .

Let  $N^* = \bigcup_{i < \omega_1} N^i$ . Then  $N^*$  has size  $\omega_1$  but omits  $p_0$ : Otherwise, there is  $a \in N^*$  realizing  $p_0$ . Since  $a \notin N^0$ , there is  $i < \omega_1$  such that  $a \in N^{i+1} \setminus N^i$ . Then  $\text{tp}(a/N^i)$  is nonalgebraic and extends  $p$ . Hence,  $\text{tp}(a/N^i) = p_i$  by Proposition 6.1 since  $N^i$  is a (super)limit over  $M$ , but this is a contradiction since  $a \in N^{i+1}$  and  $p_i$  is not realized in  $N^{i+1}$ . So,  $p_0$  is not realized in  $N^*$ , which implies that  $N^*$  is not saturated, contradicting Corollary 2.3.  $\square$

This was the last ingredient of the proof of upward categoricity. We will give the complete proof in the next chapter.

## 7. UPWARD CATEGORICITY: GOING UP INDUCTIVELY

In this chapter, we assume that  $\mathcal{K}$  is a local abstract elementary class, with amalgamation and arbitrarily large models. We assume that  $\text{LS}(\mathcal{K}) = \aleph_0$  and that  $\mathcal{K}$  is categorical in  $\aleph_1$ .

As we explained in the introduction, the idea is to prove by induction on  $\mu \geq \aleph_1$  that every model of size  $\mu$  is saturated. This implies categoricity in  $\mu$  by Theorem 2.4. The inductive framework is the reason why the assumption that all the uncountable models of size at most  $\mu$  are saturated will appear in the following two propositions.

We first show that we can extend quasiminimal types to larger models, provided all the intermediate models are saturated:

**Proposition 7.1.** *Let  $p \in S(M)$  be quasiminimal, with  $M$  countable. Let  $\mu \geq \aleph_1$  and assume that every model of size  $\kappa$  is saturated, with  $\aleph_1 \leq \kappa \leq \mu$ . Then  $p$  has a unique nonalgebraic extension to any model of size  $\mu$ .*

*Proof.* We prove inductively that there exists a unique nonalgebraic extension of  $p$  in  $S(N)$  by induction on  $\mu = \|N\| \geq \aleph_1$ . For  $\mu = \aleph_1$  this is Proposition 6.2. Now assume that  $\mu > \aleph_1$ . By assumption, we can find  $(N_i : i < \mu)$  an increasing and continuous chain of saturated models of size  $\|N_i\| = |i| + \aleph_1$ . By induction hypothesis, there exists a unique nonalgebraic  $p_i \in S(N_i)$  extending  $p$ . By uniqueness, the sequence  $(p_i : i < \mu)$  is  $\subseteq$ -increasing, so there exists  $q \in S(N)$  extending each  $p_i$  by Proposition 4.9. Now the uniqueness of  $q$  is as in Proposition 6.2, since  $N$  is saturated.  $\square$

The key to carry out the induction in the main theorem is the successor case. We use the absence of Vaughtian pairs to show this. The proof of the next proposition is a simplification of the final argument in [Sh394] and Theorem 4.1 of [GrVD].

**Proposition 7.2.** *Let  $\mu \geq \aleph_1$ . Assume that all models of size  $\kappa$  are saturated, with  $\aleph_1 \leq \kappa \leq \mu$ . Then all models of size  $\mu^+$  are saturated.*

*Proof.* By Theorem 5.5 we can find a quasiminimal  $p \in S(M)$  over a countable model  $M$ . Let  $N$  be a model of size  $\mu^+$ , with  $\mu \geq \aleph_1$ . By assumption,  $N$  is  $\mu$ -saturated, so we may assume that  $M \prec_{\mathcal{K}} N$ . Observe that since there are no  $(p, \mu)$ -Vaughtian pair by Theorem 6.5, every nonalgebraic extension of  $p$  to a submodel of  $N' \prec_{\mathcal{K}} N$  of size  $\mu$  must be realized in  $N$ , otherwise by choosing any  $N''$  of size  $\mu$ , with  $N' \prec_{\mathcal{K}} N'' \prec_{\mathcal{K}} N$ , and  $N' \neq N''$ , we have a  $(p, \mu)$ -Vaughtian pair. We now prove:

**Claim.** Let  $M \prec_{\mathcal{K}} M'$ , with  $M'$  of size  $\mu$ , and a  $\mathcal{K}$ -embedding  $f : M' \rightarrow N$  which is the identity on  $M$ . Let  $a \in \mathfrak{C}$  realize  $p$ . Then there exist  $M''$  of size  $\mu$ , with  $M' \prec_{\mathcal{K}} M''$  and  $a \in M''$ , and a  $\mathcal{K}$ -embedding  $g : M'' \rightarrow N$  extending  $f$ .

*Proof.* If  $a$  is already in  $M'$ , there is nothing to do. Otherwise the type  $p' = \text{tp}(a/M')$  is a nonalgebraic extension of  $p$ , so  $f(p')$  is a nonalgebraic extension of  $p$  over a submodel of  $N$  of size  $\mu$ . Thus  $f(p')$  must be realized by some  $b \in N$ , by the observation in the first paragraph. Choose an automorphism  $F$  of  $\mathfrak{C}$  extending  $f$  sending  $a$  to  $b$ , and choose  $N' \prec_{\mathcal{K}} N$  of size  $\mu$  containing  $b$  such that  $f(M') \prec_{\mathcal{K}} N'$ . Let  $M'' = F^{-1}(N')$  and  $g = F \upharpoonright M''$ . Then  $g : M'' \rightarrow N$  extends  $f$  and  $M''$  contains  $a$  and extends  $M'$ , as desired.  $\square$

We now show that  $N$  is saturated. Fix  $M_0 \prec_{\mathcal{K}} N$  of size  $\mu$  and  $q \in S(M_0)$ . We will show that  $q$  is realized in  $N$ . First, we may assume that  $M \prec_{\mathcal{K}} M_0$ , since  $M_0$  is saturated (and  $\mu \geq \aleph_1$ ). We construct two increasing chains of models

$$(M_n : n < \omega) \text{ and } (M'_n : n < \omega), \text{ with } M_n \prec_{\mathcal{K}} M'_n,$$

such that each model is of size  $\mu$ ,  $M_0$  is as given above,  $M'_0$  realizes  $q$ , every realization of  $p$  in  $M'_n$  is in  $M_{n+1}$ . We also construct an increasing chain of  $\mathcal{K}$ -embeddings

$$f_n : M_n \rightarrow N, \text{ such that } f_n \upharpoonright M_0 = \text{id}_{M_0}.$$

This is easy to do: Let  $M_0$  be as above, and choose  $M'_0$  of size  $\mu$  extending  $M_0$  and realizing  $q$ . Let  $f_0 = \text{id}_{M_0} : M_0 \rightarrow N$ . Now having constructed  $f_n : M_n \rightarrow N$  and  $M_n \prec_{\mathcal{K}} M'_n$ , there are at most  $\mu$  realizations of  $p$  in  $M'_n \setminus M_n$ , so by applying the previous claim  $\mu$ -times, we can find  $M_{n+1}$  of size  $\mu$  extending  $M_n$  containing every realization of  $p$  in  $M'_n$ , as well as a  $\mathcal{K}$ -embedding  $f_{n+1} : M_{n+1} \rightarrow N$ , extending  $f_n$ . Choose  $M'_{n+1}$  any model of size  $\mu$  containing  $M_{n+1} \cup M'_n$ .

This is enough: Let  $M_\omega = \bigcup_{n < \omega} M_n$  and  $M'_\omega = \bigcup_{n < \omega} M'_n$ . Then  $M_\omega \prec_{\mathcal{K}} M'_\omega$  are models of size  $\mu$ . By Proposition 7.1 there is a nonalgebraic extension of  $p$  in  $S(M_\omega)$ , and by construction, this nonalgebraic extension is not realized in  $M'_\omega \setminus M_\omega$ . Since there are no  $(p, \mu)$ -Vaughtian pairs by Theorem 6.5, this implies that  $M_\omega = M'_\omega$ . Hence,  $\bigcup_{n < \omega} f_n$  is a  $\mathcal{K}$ -embedding from  $M'_\omega$  into  $N$  fixing  $M_0$ , and so sends a realization of  $q$  in  $M'_\omega$  to a realization of  $q$  in  $N$ . This shows that  $q$  is realized in  $N$ .  $\square$

We finish this chapter with the complete proof of the upward categoricity theorem. We repeat all the assumptions for emphasis.

**Theorem 7.3.** *Let  $\mathcal{K}$  be a local abstract elementary class with amalgamation, arbitrarily large models, and countable Löwenheim-Skolem number. If  $\mathcal{K}$  is categorical in  $\aleph_1$  then  $\mathcal{K}$  is categorical in every uncountable cardinal.*

*Proof.* We prove, by induction on  $\mu \geq \aleph_1$ , that every model of size  $\mu$  is saturated. This implies categoricity in every uncountable cardinal  $\mu$  by Theorem 2.4. The

base case of  $\mu = \aleph_1$  is given by Corollary 2.3. The successor case is given by Proposition 7.2. So we are left with the limit case which is easy: If  $N$  has size  $\mu$  a limit cardinal and  $q \in S(M)$  with  $M \prec_{\mathcal{K}} N$  of size less than  $\mu$ , then we can choose  $N'$ , with  $M \prec_{\mathcal{K}} N' \prec_{\mathcal{K}} N$  of size  $\|M\|^+ < \mu$ . So  $N'$  is saturated by induction hypothesis. So  $q$  is realized in  $N'$  and hence in  $N$ . This shows that  $N$  is saturated and finishes the proof.  $\square$

## 8. APPENDIX

This chapter contains a direct argument showing that abstract elementary classes cannot ‘define’ long well-orders in the sense that if an abstract elementary class contains a model with a predicate which is well-ordered and sufficiently long, then this predicate contains an infinite descending sequence in another model. There are no amalgamation assumptions on the abstract elementary class. The argument is extracted from Shelah and Väänänen (Theorem 21 in [ShVä]). It also shows that the class of well-orders is not an abstract elementary class.

**Theorem 8.1.** *Let  $(\mathcal{K}, \prec_{\mathcal{K}})$  be an abstract elementary class in a language  $L$  containing a predicate  $P$  and a binary relation  $<$ . Suppose there is a model  $N \in \mathcal{K}$  such that  $(P^N, <^N)$  is well-ordered with order-type  $(2^{\text{LS}(\mathcal{K})})^+$ . Then there exists  $M \in \mathcal{K}$  such that  $(P^M, <^M)$  contains an infinite strictly descending sequence.*

*Proof.* Let  $\mu = 2^{\text{LS}(\mathcal{K})}$ . Fix  $N \in \mathcal{K}$  such that  $(P^N, <^N)$  is well-ordered under  $<$  of order-type  $\mu^+$ . Without loss of generality, we may assume that  $(P^N, <^N) = (\mu^+, <)$ , where  $<$  is the usual ordering on the ordinals.

The idea is to construct an  $\prec_{\mathcal{K}}$ -increasing sequence of models in  $\mathcal{K}$

$$(M_n : n < \omega)$$

and a sequence of elements  $(a_n : n < \omega)$  such that  $a_0, \dots, a_{n-1} \in M_n$  forms a strictly descending sequence in  $(P^{M_n}, <^{M_n})$ . This implies the conclusion of the theorem since  $M := \bigcup_{n < \omega} M_n \in \mathcal{K}$  and  $(a_n : n < \omega)$  forms an infinite strictly descending sequence in  $(P^M, <^M)$ .

To carry this out, we construct auxiliary models  $M_n^i$  and sequences  $\bar{b}_n^i$ , for each  $i < \mu^+$ , such that:

- (1) Each  $M_n^i \prec_{\mathcal{K}} N$  is of size  $\text{LS}(\mathcal{K})$ .
- (2) The sequence  $\bar{b}_n^i \in M_n^i$  is a strictly descending sequence of length  $n$  in  $P^N$  whose last element is greater than  $i$ .
- (3) For each  $i, k < \mu^+$  we have  $(M_n^i, \bar{b}_n^i) \cong (M_n^k, \bar{b}_n^k)$ .
- (4) If  $j$  is the last element of the sequence  $\bar{b}_{n+1}^i$  then:

$$M_n^j \prec_{\mathcal{K}} M_{n+1}^i \quad \text{and} \quad \bar{b}_{n+1}^i = \bar{b}_n^j \hat{\ } \langle j \rangle.$$

Note that there is no ambiguity in the phrasing of (2) since  $M_n^i \subseteq N$  by (1) and  $(P^N, <^N) = (\mu^+, <)$ .

We do this by induction on  $n$  for all  $i < \mu^+$ . For  $n = 0$ , choose  $M_0^i := M_0 \prec_{\mathcal{K}} N$  of size  $\text{LS}(\mathcal{K})$  and let  $\bar{b}_0^i$  be the empty sequence. Now having constructed  $M_n^i, \bar{b}_n^i$ , for each  $i < \mu^+$ , choose  $N_n^i \prec_{\mathcal{K}} N$  containing  $M_n^i \cup \{i\}$  of size  $\text{LS}(\mathcal{K})$  (recall that  $i \in P^N = \mu^+$ ). Notice that  $M_n^i \prec_{\mathcal{K}} N_n^i$ . Now the sequence  $\bar{b}_n^i \hat{\ } \langle i \rangle$  is strictly descending by induction hypothesis (2). Since there are at most

$\mu = 2^{\text{LS}(\mathcal{K})}$  nonisomorphic models of size  $\text{LS}(\mathcal{K})$  there exists  $S \subseteq \mu^+$  of size  $\mu^+$  such that

$$(N_n^i, \bar{b}_n^i, i) \cong (N_n^k, \bar{b}_n^k, k), \quad \text{for each } i, k \in S.$$

Let  $j(i) := \min\{j : j \in S \text{ and } j > i\}$ , for each  $i < \mu^+$ . Now let  $M_{n+1}^i := N_n^{j(i)}$  and  $\bar{b}_{n+1}^i := \bar{b}_n^{j(i) \wedge \langle j(i) \rangle}$ .

It is immediate that (1)–(3) are satisfied. For (4), observe that if  $j$  is the last element of  $\bar{b}_{n+1}^i$  then  $j = j(i)$  and we have

$$M_n^j \prec_{\mathcal{K}} N_n^j = M_{n+1}^i.$$

This completes the auxiliary construction.

We now finish the theorem by constructing  $M_n$  and  $a_0, \dots, a_{n-1}$ , by induction on  $n < \omega$ , with the additional inductive requirement that there is an isomorphism

$$f_n : (M_n^0, \bar{b}_n^0) \cong (M_n, a_0, \dots, a_{n-1}).$$

For  $n = 0$  simply let  $M_0 := M_0^0$  and let  $f_0$  be the identity. Now assume that  $M_n, a_0, \dots, a_{n-1}$ , and  $f_n$  have been constructed. Let  $j$  be the last element of the sequence  $\bar{b}_{n+1}^0$ . By (3) there is an isomorphism  $g : (M_n^0, \bar{b}_n^0) \cong (M_n^j, \bar{b}_n^j)$ , so we have

$$f_n \circ g^{-1} : (M_n^j, \bar{b}_n^j) \cong (M_n, a_0, \dots, a_{n-1}).$$

By (4) above we have  $M_n^j \prec_{\mathcal{K}} M_{n+1}^0$ , so we can extend  $f_n \circ g^{-1}$  to an  $L$ -embedding  $f_{n+1}$  with domain all of  $M_{n+1}^0$ . We let  $M_{n+1}$  be the image of  $M_{n+1}^0$  under  $f_{n+1}$  and let  $a_n := f_{n+1}(j)$ . Then  $M_n \prec_{\mathcal{K}} M_{n+1}$  since  $M_n^j \prec_{\mathcal{K}} M_{n+1}^0$  and  $\prec_{\mathcal{K}}$  is preserved by the isomorphism  $f_{n+1}$ . Furthermore,

$$f_{n+1} : (M_{n+1}^0, \bar{b}_{n+1}^0) \cong (M_{n+1}, a_0, \dots, a_{n-1}, a_n).$$

This finishes the proof.  $\square$

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